

TOEPLITZ OPERATORS AND HAMILTONIAN TORUS ACTIONS

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ABSTRACT. This paper is devoted to semi-classical aspects of symplectic reduction. Consider a compact prequantizable Kähler manifold M with a Hamiltonian torus action. In the seminal paper [13], Guillemin and Sternberg introduced an isomorphism between the invariant part of the quantum space associated to M and the quantum space associated to the symplectic quotient of M , provided this quotient is non-singular. We prove that this isomorphism is a Fourier integral operator and that the Toeplitz operators of M descend to Toeplitz operators of the reduced phase space. We also extend these results to the case where the symplectic quotient is an orbifold and estimate the spectral density of a reduced Toeplitz operator, a result related to the Riemann-Roch-Kawasaki theorem.

1. INTRODUCTION

Consider a symplectic manifold (M, ω) with a Hamiltonian action of a d -dimensional torus \mathbb{T}^d . Let μ be a momentum map. Following Marsden-Weinstein [18], if λ is a regular value of μ , the reduced space

$$M_r := \mu^{-1}(\lambda)/\mathbb{T}^d$$

is naturally endowed with a symplectic form ω_r . The quantum analogue of this reduction has been the subject of important studies, starting from the paper [13] of Guillemin and Sternberg, and has led to many versions of the “quantization commutes with reduction” theorem. In most of these articles, the quantization is defined as a Riemann-Roch number or the index of a spin-c Dirac operator which represents the dimension of a virtual quantum space, cf. the review article [23]. The relationships between deformation quantization and symplectic reduction have also been considered [26], [11].

This paper is devoted to the quantum aspects of symplectic reduction in the semi-classical setting. Here the quantization consists of a Hilbert space with a semi-classical algebra of operators. More precisely, we assume that M is compact, Kähler and endowed with a prequantization bundle $L \rightarrow M$, i.e. a Hermitian line bundle with a connection of curvature $-i\omega$. For every positive integer k , let us define the quantum space \mathcal{H}_k as the space of holomorphic sections of L^k . The semi-classical limit is $k \rightarrow \infty$ and the operators we will consider are the Toeplitz operators, introduced by Berezin in [2]. The application of microlocal techniques in this context started with Boutet de Monvel and Guillemin [4]. This point of view made it possible to extend many results known for the pseudodifferential operators with small parameter to the Toeplitz operators, as for instance, the trace formula [3] and the Bohr-Sommerfeld conditions [7].

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Assume that the torus action preserves the complex structure of M . Then following Kostant and Souriau we can associate to the components (μ_1, \dots, μ_d) of the moment map μ some commuting operators $M_1, \dots, M_d : \mathcal{H}_k \rightarrow \mathcal{H}_k$. Suppose that $\lambda = (\lambda_1, \dots, \lambda_d)$ is a joint eigenvalue of these operators when $k = 1$ and that the torus action on $\mu^{-1}(\lambda)$ is free. The following theorem is a slight reformulation of the main result of Guillemin and Sternberg.

Theorem 1.1 ([13]). *M_r inherits a natural Kähler structure and a prequantization bundle L_r , which defines quantum spaces $\mathcal{H}_{r,k}$. Furthermore, for any k , there exists a natural vector space isomorphism V_k from*

$$\mathcal{H}_{\lambda,k} := \cap_{i=1}^d \text{Ker}(M_i - \lambda_i)$$

onto $\mathcal{H}_{r,k}$.

The various quantum spaces have natural scalar products induced by the Hermitian structure of the prequantization bundles and the Liouville measures, but unfortunately the isomorphism V_k is not necessarily unitary. So we will use

$$U_k := V_k(V_k^* V_k)^{-\frac{1}{2}} : \mathcal{H}_{\lambda,k} \rightarrow \mathcal{H}_{r,k}$$

instead. Our first result relates the Toeplitz operators of M with the Toeplitz operators of M_r .

Theorem 1.2. *Let $(T_k : \mathcal{H}_k \rightarrow \mathcal{H}_k)_{k \in \mathbb{N}^*}$ be a Toeplitz operator of M which commutes with M_1, \dots, M_d , and with principal symbol $f \in C^\infty(M)$. Then*

$$(U_k T_k U_k^* : \mathcal{H}_{r,k} \rightarrow \mathcal{H}_{r,k})_k$$

is a Toeplitz operator of M_r . Furthermore, f is \mathbb{T}^d -invariant and the principal symbol f_r of $(U_k T_k U_k^)$ is such that $p^* f_r = j^* f$, where p and j are respectively the projection $\mu^{-1}(\lambda) \rightarrow M_r$ and the embedding $\mu^{-1}(\lambda) \rightarrow M$.*

When the torus action on $\mu^{-1}(\lambda)$ is not free but locally free, the reduced space M_r is not a manifold, but an orbifold. These spaces with finite quotient singularities were first introduced by Satake in [22]. Many results or notions of differential geometry have been generalized to orbifolds: index theorem [17], fundamental group [24], string theory [21]. Not surprisingly, theorems 1.1 and 1.2 are still valid in this case. Motivated by this, we prove the basic properties of the Toeplitz operators on the orbifold M_r : description of their Schwartz kernel and the symbolic calculus. Our second main result is the estimate of the spectral density of a Toeplitz operator on the orbifold M_r . This simple result in the manifold case involves here oscillatory contribution of the inertia orbifolds or twisted sectors associated to M_r and is related to the Kawasaki-Riemann-Roch theorem [17], cf. theorems 2.3 and 6.9 for precise statements.

With a view towards application, we also consider the simple case where M is \mathbb{C}^n with a linear circle action whose momentum map is a proper harmonic oscillator. The quantum data associated with \mathbb{C}^n are defined by the Bargmann representation and the reduced space is a twisted projective space. Actually, this is nearly a particular case of the previous setting, except that \mathbb{C}^n is not compact. As a corollary of theorem 1.2, the spectral analysis of an operator commuting with the quantum harmonic oscillator is reduced to that of a Toeplitz operator on a projective space. In collaboration with San Vu Ngoc, we plan to apply this to the semi-excited spectrum of a Schrödinger operator with a non-degenerate potential well.

We also prove that the isomorphism V_k of theorem 1.1 and its unitarization U_k are Fourier integral operators. Thus we can interpret theorem 1.2 as a composition of Fourier integral operators with underlying compositions of canonical relations. Actually our proof of theorem 1.2 is elementary in the sense that it relies on the

geometric properties of the isomorphism V_k and doesn't use the usual tools of microlocal analysis. But with the more general point of view of Fourier integral operators, we hope that we can extend the "quantization commutes with reduction" theorems by using microlocal techniques. For instance, theorems 1.1 and 1.2 should hold with general Toeplitz operators M_i whose joint principal symbol define a momentum map without assuming that the action preserves the complex structure. Also the Kähler structure is certainly not necessary. This microlocal approach is also related to another paper of Guillemin and Sternberg [14] (cf. sections 4.3 and 5.3 for a comparison with our results).

The organization of the paper is as follows. Section 2 contains detailed statements of our main results for the harmonic oscillator on \mathbb{C}^n . In section 3, we introduce our set-up in the compact Kähler case and recall the results of [13] proving that the reduced quantum space is isomorphic to the joint eigenspace. Section 4 contains the statements and proofs of our main results for the reduction of Toeplitz operators. In section 5, we interpret these results as compositions of Fourier integral operators. Sections 6 and 7 are devoted to the Toeplitz operators on Kähler orbifolds.

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2. STATEMENT OF THE RESULTS FOR THE HARMONIC OSCILLATOR

Assume \mathbb{C}^n is endowed with the usual symplectic 2-form $\omega = i(dz_1 \wedge d\bar{z}_1 + \dots + dz_n \wedge d\bar{z}_n)$. Let H be the harmonic oscillator

$$H := \mathfrak{p}_1 |z_1|^2 + \dots + \mathfrak{p}_n |z_n|^2$$

where $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are positive relatively prime integers. Consider the scalar product

$$(1) \quad (\Psi, \Psi')_{\mathbb{C}^n} = \int_{\mathbb{C}^n} e^{-\hbar^{-1}|z|^2} \Psi(z) \cdot \bar{\Psi}'(z) \frac{|dz \cdot d\bar{z}|}{n!}$$

where Ψ, Ψ' are functions on \mathbb{C}^n and $|z|^2 = |z_1|^2 + \dots + |z_n|^2$. The Bargmann space \mathcal{H} is the Hilbert space of holomorphic functions Ψ on \mathbb{C}^n such that $(\Psi, \Psi)_{\mathbb{C}^n} < \infty$. The quantum harmonic oscillator is the unbounded operator of \mathcal{H}

$$(2) \quad H := \hbar(\mathfrak{p}_1 z_1 \partial_{z_1} + \dots + \mathfrak{p}_n z_n \partial_{z_n})$$

with domain the space of polynomials on \mathbb{C}^n .

2.1. Symplectic reduction. The Hamiltonian flow of H induces an action of S^1 on the level set $P := \{H = 1\}$

$$S^1 \times P \rightarrow P, \quad \theta, z \rightarrow l_\theta \cdot z = (z_1 e^{i\theta \mathfrak{p}_1}, \dots, z_n e^{i\theta \mathfrak{p}_n}) \text{ if } z = (z_1, \dots, z_n).$$

Define the reduced space M_r as the quotient P/S^1 . If $\mathfrak{p}_1 = \dots = \mathfrak{p}_n = 1$, the action is free, M_r is a manifold and the projection $P \rightarrow M_r$ is the Hopf fibration. When the \mathfrak{p}_i are not all equal to 1, the action is not free, but locally free. Hence M_r is not a manifold, but an orbifold. In any cases, M_r is naturally endowed with a symplectic 2-form ω_r .

We may also define a complex structure on the space M_r by viewing it as a complex quotient. Consider the holomorphic action of \mathbb{C}^*

$$(3) \quad \mathbb{C}^* \times \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad u, (z_1, \dots, z_n) \rightarrow (z_1 u^{\mathfrak{p}_1}, \dots, z_n u^{\mathfrak{p}_n})$$

Each \mathbb{C}^* -orbit of $\mathbb{C}^n - \{0\}$ intersects P in a S^1 -orbit, which identifies M_r with $\mathbb{C}^n - \{0\}/\mathbb{C}^*$. This quotient is called a twisted projective space, the standard projective space is obtained when $\mathfrak{p}_1 = \dots = \mathfrak{p}_n = 1$. The complex structure is compatible with the symplectic form ω_r . So M_r is a Kähler orbifold.

2.2. Quantum Reduction. H has a discrete spectrum given by

$$\text{Sp}(H) = \{\hbar(\mathbf{p}_1\alpha(1) + \dots + \mathbf{p}_n\alpha(n)); \alpha \in \mathbb{N}^n\}$$

Hence 1 is an eigenvalue only if \hbar is of the form $1/k$ with k a positive integer. Since we will only consider the eigenvalue 1, we assume from now on that

$$\hbar = 1/k \text{ with } k \in \mathbb{N}^*$$

and use the large parameter k instead of the small parameter \hbar . We denote by \mathcal{H}_k and H_k the Bargmann space and the quantum harmonic oscillator. The vector space

$$\mathcal{H}_{1,k} := \text{Ker}(H_k - 1)$$

is generated by the monomials z^α such that $\mathbf{p}_1\alpha(1) + \dots + \mathbf{p}_n\alpha(n) = k$. So a state $\Psi \in \mathcal{H}_k$ belongs to $\mathcal{H}_{1,k}$ if and only if it is invariant in the sense that

$$\Psi(z_1 u^{\mathbf{p}_1}, \dots, z_n u^{\mathbf{p}_n}) = u^k \Psi(z_1, \dots, z_n).$$

Hence there is a holomorphic line orbi-bundle $L_r \rightarrow M_r$ such that $\mathcal{H}_{1,k}$ identifies with the space $\mathcal{H}_{r,k}$ of holomorphic sections of L_r^k . L_r has a natural Hermitian structure and connection of curvature $-i\omega_r$, which turns it into a prequantization orbi-bundle (cf. section 3). We denote by V_k the isomorphism from $\mathcal{H}_{1,k}$ to $\mathcal{H}_{r,k}$

2.3. Reduction of the operators. On the Bargmann space a usual way to define operators is the Wick or Toeplitz quantization. Denote by Π_k the orthogonal projector of $L^2(\mathbb{C}^n, e^{-k|z|^2} |dz.d\bar{z}|)$ onto \mathcal{H}_k . To every function f of \mathbb{C}^n we associate the operator $\text{Op}(f)$ of \mathcal{H}_k defined by

$$\text{Op}(f) : \Psi \rightarrow \Pi_k(f.\Psi)$$

More generally, we consider multiplicators f which depend on k . Define the class $S(\mathbb{C}^n)$ of symbols $f(., k)$ which are sequences of $C^\infty(\mathbb{C}^n)$ satisfying

- there exists $C > 0$ and N such that $|f(z, k)| \leq C(1 + |z|)^N$, $\forall z \in \mathbb{C}^n$, $\forall k$.
- $f(., k)$ admits an asymptotic expansion of the form

$$\sum_{l=0}^{\infty} k^{-l} f_l + O(k^{-\infty})$$

with $f_0, f_1, \dots \in C^\infty(\mathbb{C}^n)$ for the C^∞ topology on a neighborhood of P .

For such a symbol, we consider $\text{Op}(f(., k))$ as an unbounded operator of \mathcal{H}_k with domain polynomials on \mathbb{C}^n . Its *principal symbol* is the function f_0 . If $f(., k)$ is invariant with respect to the Hamiltonian flow of H , then $\text{Op}(f(., k))$ sends $\mathcal{H}_{1,k}$ into itself.

Remark 2.1. The class of Toeplitz operators with symbol in $S(\mathbb{C}^n)$ contains the algebra of differential operators generated by $\frac{1}{k}\partial_{z_i}$ and z_i . Indeed, let

$$f(., k) = P_0 + k^{-1}P_1 + \dots + k^{-M}P_M$$

where $P_0(\bar{z}, z), \dots, P_M(\bar{z}, z)$ are polynomials of $\mathbb{C}[\bar{z}, z]$. Then $\text{Op}(f(., k))$ is the operator

$$P_0(\frac{1}{k}\partial_z, z) + k^{-1}P_1(\frac{1}{k}\partial_z, z) + \dots + k^{-M}P_M(\frac{1}{k}\partial_z, z).$$

Its principal symbol is P_0 . If the P_i are linear combinations of the monomials $\bar{z}^\alpha z^\beta$ such that $\langle \mathbf{p}, \alpha - \beta \rangle = 0$, then $f(., k)$ Poisson commutes with H and $\text{Op}(f(., k))$ preserves the eigenspaces of the quantum harmonic oscillator. \square

Let U_k be the unitary map $V_k(V_k^* V_k)^{-\frac{1}{2}} : \mathcal{H}_{1,k} \rightarrow \mathcal{H}_{r,k}$, that we extend to \mathcal{H}_k in such a way that it vanishes on the orthogonal space to $\mathcal{H}_{1,k}$. The *reduced* operator of $\text{Op}(f(., k))$ is the operator

$$U_k \text{Op}(f(., k)) U_k^* : \mathcal{H}_{r,k} \rightarrow \mathcal{H}_{r,k}$$

Our main result says it is a Toeplitz operator.

Theorem 2.2. *Let $f(., k)$ be a symbol of $S(\mathbb{C}^n)$. Then there exists a sequence $g(., k)$ of $C^\infty(M_r)$, which admits an asymptotic expansion of the form $\sum_{l=0}^\infty k^{-l} g_l + O(k^{-\infty})$ for the C^∞ topology, such that*

$$U_k \text{Op}(f(., k)) U_k^* = \Pi_{r,k} g(., k) + O(k^{-\infty})$$

where $\Pi_{r,k}$ is the orthogonal projector onto $\mathcal{H}_{r,k}$ and the $O(k^{-\infty})$ is for the uniform norm. Furthermore, the principal symbol g_0 of the reduced operator is given by

$$g_0(p(x)) = \int_{S^1} f_0(l_\theta.x) \frac{|d\theta|}{2\pi}, \quad \forall x \in P$$

where p is the projection $P \rightarrow M_r$ and f_0 is the principal symbol of $\text{Op}(f(., k))$.

2.4. Spectral density. Consider a self-adjoint Toeplitz operator $(T_k)_k$ of M_r ,

$$T_k = \Pi_{r,k} g(., k) + O(k^{-\infty}) : \mathcal{H}_{r,k} \rightarrow \mathcal{H}_{r,k}$$

where $g(., k)$ is a sequence of $C^\infty(M_r, \mathbb{R})$ with an asymptotic expansion of the form $\sum_{l=0}^\infty k^{-l} g_l + O(k^{-\infty})$ in the C^∞ topology. Let d_k be the dimension of $\mathcal{H}_{r,k}$ and

$$\lambda_1(k) \leq \lambda_2(k) \leq \dots \leq \lambda_{d_k}(k)$$

be the eigenvalues of T_k counted with multiplicity.

Let $f \in C^\infty(\mathbb{R})$. The estimate of $\sum_{i=1}^{d_k} f(\lambda_i(k))$ as $k \rightarrow \infty$ is a standard semi-classical result when M_r is a manifold. In the orbifold case, this result involves the singular locus of M_r . Denote by G the set of $\zeta = e^{i\theta}$ such that

$$P_\zeta := \{z \in P; l_\theta.z = z\}$$

is not empty. A straightforward computation leads to

$$G = \{\zeta \in \mathbb{C}^*; \zeta^{p_i} = 1 \text{ for some } i\}$$

and

$$P_\zeta = \mathbb{C}_\zeta \cap P \text{ with } \mathbb{C}_\zeta = \{z \in \mathbb{C}^n; z_i = 0 \text{ if } \zeta^{p_i} \neq 1\}.$$

The Hamiltonian flow of H preserves P_ζ . Let M_ζ be the quotient of P_ζ by the induced S^1 -action. It is a twisted projective space which embeds into M_r as a symplectic suborbifold. Denote by $n(\zeta)$ its complex dimension. Finally, let $m(\zeta)$ be the greatest common divisor of $\{p_i; \zeta^{p_i} = 1\}$.

Theorem 2.3. *For every function $f \in C^\infty(\mathbb{R})$,*

$$\sum_{i=1}^{d_k} f(\lambda_i(k)) = \sum_{\zeta \in G} \left(\frac{k}{2\pi}\right)^{n(\zeta)} \zeta^{-k} \sum_{l=0}^\infty k^{-l} I_l(\zeta) + O(k^{-\infty})$$

The leading coefficients are given by

$$I_0(\zeta) = \frac{1}{m(\zeta)} \left(\prod_{i; \zeta^{p_i} \neq 1} (1 - \zeta^{p_i})^{-1} \right) \int_{M_\zeta} f(g_0) \delta_{M_\zeta},$$

where δ_{M_ζ} is the Liouville measure of M_ζ .

Observe that $M_1 = M_r$. The other M_ζ are of positive codimension and are the closures of the singular stratas of M_r . Hence at first order, the formula is the same as in the manifold case

$$\sum_{i=1}^{d_k} f(\lambda_i) = \left(\frac{k}{2\pi}\right)^{n-1} \int_{M_r} f(g_0) \delta_{M_r} + O(k^{n-2})$$

Furthermore applying this result with $f \equiv 1$, we obtain an estimate of the dimension of \mathcal{H}_k . When k is sufficiently large, this dimension is also given by the Riemann-Roch-Kawasaki theorem and both results are in agreement (cf. remark 6.11).

3. THE GUILLEMIN-STERNBERG ISOMORPHISM

Let M be a compact connected Kähler manifold. Denote by $\omega \in \Omega^2(M, \mathbb{R})$ the fundamental two-form. Assume that M is endowed with a prequantization bundle $L \rightarrow M$, that is L is a Hermitian line bundle with a connection of curvature $-i\omega$. (M, ω) is a symplectic manifold and represents the classical phase space. For every positive integer k define the quantum space \mathcal{H}_k as the space of holomorphic sections of $L^k \rightarrow M$.

Assume that M is endowed with an effective Hamiltonian torus action

$$(4) \quad \mathbb{T}^d \times M \rightarrow M, \quad \theta, x \rightarrow l_\theta.x$$

which preserves the complex structure. Let \mathfrak{t}_d be the Lie algebra of \mathbb{T}^d . If $\xi \in \mathfrak{t}_d$, we denote by $\xi^\#$ the associated vector field of M . Let

$$\mu : M \rightarrow \mathfrak{t}_d^*$$

be the moment map, so $\omega(\xi^\#, \cdot) + d\langle \mu, \xi \rangle = 0$.

Following Kostant and Souriau, for every $\xi \in \mathfrak{t}_d$ and every positive integer k we define the operator $M_{\xi,k}$:

$$M_{\xi,k} := \langle \mu, \xi \rangle + \frac{1}{ik} \nabla_{\xi^\#} : \mathcal{H}_k \rightarrow \mathcal{H}_k.$$

It has to be considered as the quantization of the classical observable $\langle \mu, \xi \rangle \in C^\infty(M)$. Since the Poisson bracket of $\langle \mu, \xi \rangle$ and $\langle \mu, \xi' \rangle$ vanishes, one proves that $M_{\xi,k}$ commutes with $M_{\xi',k}$. The *joint spectrum* of the $M_{\xi,k}$ is the set of covectors $\lambda \in \mathfrak{t}_d^*$ such that

$$\mathcal{H}_{\lambda,k} := \{\Psi \in \mathcal{H}_k; \quad M_{\xi,k}\Psi = \langle \lambda, \xi \rangle \Psi, \quad \forall \xi \in \mathfrak{t}_d\}$$

is not reduced to (0) .

The joint eigenvalues are related to the values of μ in the following way. First, recall the convexity theorem of Atiyah [1] and Guillemin-Sternberg [12]: the image under μ of the fixed point set of M is a finite set

$$\{\nu_1, \dots, \nu_s\}$$

and $\mu(M)$ is the convex hull of this set.

Theorem 3.1. *Let ν be a value of μ at some fixed point. Let $(\lambda, k) \in \mathfrak{t}_d^* \times \mathbb{N}^*$. Then λ belongs to the joint spectrum of the $M_{\xi,k}$ only if*

$$(5) \quad \lambda \in \mu(M) \cap \left(\nu + \frac{2\pi}{k}K\right)$$

where K is the integer lattice of \mathfrak{t}_d^* .

The condition (5) doesn't depend on the choice of ν : since (M, ω) is endowed with a prequantization bundle, it is known that for every i, j

$$(6) \quad \nu_i - \nu_j \in 2\pi K.$$

That $\lambda \in \mu(M)$ is necessary has been proved by Guillemin and Sternberg (cf. theorem 5.3 of [13]). The second condition, $\lambda \in \nu + 2\pi k^{-1}K$, is an exact Bohr-Sommerfeld condition, which follows from the theory of Kostant and Souriau.

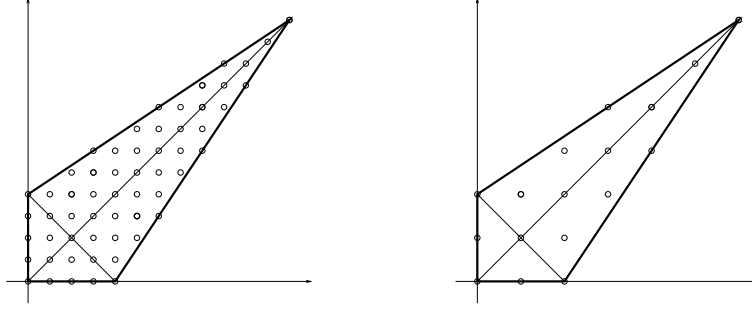
Example 3.2. Let M be the projective space \mathbb{CP}^3 with ω the Fubini-Study form

$$\omega = -i\partial\bar{\partial}(|z_1|^2 + \dots + |z_4|^2), \quad [z_1, \dots, z_4] \in \mathbb{CP}^3$$

and L the tautological bundle. Consider the torus action

$$(\theta_1, \theta_2), [z_1, \dots, z_4] \rightarrow [z_1 e^{-2i\pi(\theta_1 + \theta_2)}, z_2 e^{-6i\pi\theta_1}, z_3 e^{-6i\pi\theta_2}, z_4]$$

with momentum map $\mu = \frac{2\pi}{|z|^2}(|z_1|^2 + 3|z_2|^2, |z_1|^2 + 3|z_3|^2)$. The points on the figure



are the λ satisfying condition (5) with $k = 4$ on the left and $k = 2$ on the right. The lines are the critical values of μ . \square

Let $(\lambda, k) \in \mathfrak{t}_d^* \times \mathbb{N}^*$. Assume that (λ, k) satisfies condition (5) and that λ is a regular value of μ . Denote by P the level set $\mu^{-1}(\lambda)$. It is known that P is connected ([12], [1]). The torus action restricts to a locally free action on P . So the quotient M_r of P is a compact connected orbifold. It is naturally endowed with a symplectic form ω_r .

Theorem 3.3 (Guillemin-Sternberg [13]). *M_r inherits by reduction a Kähler structure with fundamental 2-form ω_r and a prequantization orbi-bundle $L_r^k \rightarrow M_r$ with curvature $-ik\omega_r$. Furthermore, there exists a natural isomorphism of vector space*

$$V_k : \mathcal{H}_{\lambda,k} \rightarrow \mathcal{H}_{r,k}$$

where $\mathcal{H}_{r,k}$ is the space of holomorphic sections of L_r^k .

Now consider a fixed regular value λ of μ such that the set of integers k satisfying condition (5) is not empty. Then there exists a positive integer κ such that (λ, k) satisfies (5) if and only if k is a positive multiple of κ . Furthermore the Kähler structure of M_r doesn't depend on k and for every such k ,

$$L_r^k = (L_r^\kappa)^{\otimes k/\kappa}.$$

If M_r is a manifold, it follows from Kodaira vanishing theorem and Riemann-Roch theorem that

$$(7) \quad \dim \mathcal{H}_{r,k} = \left(\frac{k}{2\pi} \right)^{n_r} \text{Vol}(M_r) + O(k^{n_r-1})$$

as k goes to infinity, where $\text{Vol}(M_r)$ is the symplectic volume of M_r and n_r its dimension. This gives a partial converse to theorem 3.1: if k is a sufficiently large positive multiple of κ , the eigenspace $\mathcal{H}_{\lambda,k}$ is not reduced to (0) , so λ belongs to the joint spectrum of the $M_{\xi,k}$.

If M_r is an orbifold, the same result holds and follows from Riemann-Roch-Kawasaki theorem [17]. Indeed, we assumed that the torus action is effective. This implies that its restriction to P is also effective. So M_r is a reduced orbifold or equivalently its principal stratum has multiplicity one. This explains why formula (7) remains unchanged, without a sum of oscillatory terms.

All the semi-classical results we prove in this paper are in this regime,

λ is fixed and $k \rightarrow \infty$ running through the set of positive multiples of κ .

In the remainder of this section, we recall the main steps of the proof of the Guillemin-Sternberg theorem. We follow the presentation given by Duistermaat in [9], that is we consider separately the reduction of the symplectic and prequantum data and the reduction of the complex structure. We explain in remarks how the same constructions apply to the harmonic oscillator.

Remark 3.4. (harmonic oscillator). The Bargmann space can be viewed as a space of holomorphic sections of a prequantization bundle over \mathbb{C}^n . Let $L := \mathbb{C}^n \times \mathbb{C}$ be the trivial bundle over \mathbb{C}^n . We identify the sections of L^k with the functions on \mathbb{C}^n . Introduce a connection and a Hermitian structure on L^k by setting

$$\nabla \Psi = d\Psi - k\Psi(\bar{z}_1 dz_1 + \dots + \bar{z}_n dz_n), \quad (\Psi, \Psi)(z) = e^{-k|z|^2} |\Psi(z)|^2.$$

In this way L^k becomes a prequantization bundle with curvature $-ik\omega$. The scalar product defined in (1) is

$$(\Psi, \Psi)_{\mathbb{C}^n} = \int_{\mathbb{C}^n} (\Psi, \Psi)(z) \frac{|dz, d\bar{z}|}{n!}$$

So the Bargmann space \mathcal{H}_k is the Hilbert space of holomorphic sections Ψ of L^k such that $(\Psi, \Psi)_{\mathbb{C}^n}$ is finite. Furthermore it is easily checked that the quantum harmonic oscillator H_k defined in (2) is given by

$$H_k \Psi = (H + \frac{1}{ik} \nabla_{X_H}) \Psi$$

where X_H is the Hamiltonian vector field of H . □

3.1. Reduction of the symplectic and prequantum data. We first lift the torus action (4). If $z \in L_x^k$ and $\xi \in \mathfrak{t}_d$, we denote by $\mathcal{T}_\xi.z$ the parallel transport of z along the path

$$[0, 1] \rightarrow M, \quad s \rightarrow l_{\exp(s\xi)}.x$$

If ν is the value of μ at some fixed point and ξ belong to the integer lattice of \mathfrak{t}_d ,

$$e^{ik\langle \nu - \mu, \xi \rangle} \mathcal{T}_\xi.z = z, \text{ for every } z \in L^k.$$

Indeed, this is obviously true if $z \in L_x^k$ where x is a fixed point and $\mu(x) = \nu$. By proposition 15.3 of [10], the result follows for every z .

Consider now $(\lambda, k) \in \mathfrak{t}_d^* \times \mathbb{N}^*$ which satisfies condition (5). Then the action of \mathbb{T}^d on M lifts to L^k

$$\mathbb{T}^d \times L^k \rightarrow L^k, \quad (\theta, z) \rightarrow \mathcal{L}_\theta.z := e^{ik\langle \lambda - \mu, \xi \rangle} \mathcal{T}_\xi.z$$

with $\xi \in \mathfrak{t}_d$ such that $\exp \xi = \theta$. One can check the following facts: for every θ , \mathcal{L}_θ is an automorphism of the prequantization bundle L^k , it preserves the complex structure. Furthermore, the obtained representation of \mathbb{T}^d on \mathcal{H}_k induces the representation of the Lie algebra \mathfrak{t}_d given by the operators

$$\nabla_{\xi^\#} + ik\langle \mu - \lambda, \xi \rangle = ik(M_{\xi, k} - \langle \lambda, \xi \rangle), \quad \xi \in \mathfrak{t}_d.$$

Hence the joint eigenspace $\mathcal{H}_{\lambda, k}$ is the space of invariant holomorphic sections

$$\mathcal{H}_{\lambda, k} = \{ \Psi \in \mathcal{H}_k; \mathcal{L}_\theta^* \Psi = \Psi, \forall \theta \in \mathbb{T}^d \}.$$

Denote by $j : P \rightarrow M$ and $p : P \rightarrow M_r$ the natural embedding and projection. Recall that the reduced symplectic 2-form ω_r is defined by $p^* \omega_r = j^* \omega$. Let L_r^k be the quotient of $j^* L^k$ by the torus action. This is an Hermitian orbi-bundle over M_r and $p^* L_r^k$ is naturally isomorphic with $j^* L^k$. Furthermore L_r^k admits a connection ∇ such that

$$p^* \nabla = j^* \nabla.$$

Its curvature is $-ik\omega_r$. So L_r^k is a prequantization orbi-bundle. Since the sections of $\mathcal{H}_{\lambda, k}$ are invariant, their restrictions to P descend to M_r ,

$$(8) \quad \mathcal{H}_{\lambda, k} \rightarrow C^\infty(M_r, L_r^k), \quad \Psi \mapsto \Psi_r \text{ such that } p^* \Psi_r = j^* \Psi.$$

This is the first definition of the Guillemin-Sternberg isomorphism. In the case the action is not free and M_r is an orbifold, more details will be given in remark 3.8.

Remark 3.5. (harmonic oscillator). As in section 2, we only consider the eigenvalue $\lambda = 1$. The lift of the S^1 -action is explicitly given by

$$S^1 \times (\mathbb{C}^n \times \mathbb{C}) \rightarrow \mathbb{C}^n \times \mathbb{C}, \quad \theta, (z_1, \dots, z_n, v) \rightarrow (e^{ip_1\theta} z_1, \dots, e^{ip_n\theta} z_n, e^{ik\theta} v).$$

$\mathcal{H}_{1,k}$ consists of the invariant holomorphic sections of L^k . If Ψ is such a section, one can check by direct computations that (Ψ, Ψ) and $\nabla \Psi$ are invariant and that $\nabla_{X_H} \Psi$ vanishes over the level set $P := \{H = 1\}$. So the quotient L_r^k of $L^k|_P$ by S^1 inherits a structure of prequantization bundle with curvature $-ik\omega_r$. \square

3.2. Complex reduction. Let $\mathbb{T}_{\mathbb{C}}^d$ be the complex Lie group $\mathbb{T}^d \oplus i\mathfrak{t}_d$ with Lie algebra $\mathfrak{t}_d \oplus i\mathfrak{t}_d$. We consider \mathbb{T}^d as a subgroup of $\mathbb{T}_{\mathbb{C}}^d$. Since the torus action preserves the complex structure of M , it can be extended in a unique way to a holomorphic action of $\mathbb{T}_{\mathbb{C}}^d$

$$(\mathbb{T}^d \oplus i\mathfrak{t}_d) \times M \rightarrow M, \quad (\theta + it), x \rightarrow l_{\theta+it}.x$$

To do this, for every $\xi \in \mathfrak{t}_d$, we define the infinitesimal generator of $i\xi$ as $J\xi^\#$, where J is the complex structure of M . Since M is compact, we can integrate $J\xi^\#$. Then one can check that this defines a holomorphic action.

In a similar way, the action on L^k extends to a holomorphic action

$$(\mathbb{T}^d \oplus i\mathfrak{t}_d) \times L^k \rightarrow L^k, \quad (\theta + it), z \rightarrow \mathcal{L}_{\theta+it}.z$$

where $\mathcal{L}_{\theta+it}$ is an automorphism of complex bundle which lifts $l_{\theta+it}$. This gives a representation of $\mathbb{T}_{\mathbb{C}}^d$ on \mathcal{H}_k . The induced representation of the Lie algebra $\mathfrak{t}_d \oplus i\mathfrak{t}_d$ is given by the operators

$$(9) \quad \nabla_{\xi^\# + J\eta^\#} + ik\langle \mu - \lambda, \xi + i\eta \rangle, \quad \xi + i\eta \in \mathfrak{t}_d \oplus i\mathfrak{t}_d.$$

Furthermore $\mathcal{H}_{\lambda,k}$ is the space of $\mathbb{T}_{\mathbb{C}}^d$ -invariant holomorphic sections.

Remark 3.6. (harmonic oscillator). The holomorphic action of \mathbb{C}^* was given in (3). It lifts to

$$\mathbb{C}^* \times (\mathbb{C}^n \times \mathbb{C}) \rightarrow \mathbb{C}^n \times \mathbb{C}, \quad u, (z_1, \dots, z_n, v) \rightarrow (u^{p_1} z_1, \dots, u^{p_n} z_n, u^k v).$$

The invariant sections of \mathcal{H}_k are obviously the sections of $\mathcal{H}_{1,k}$. \square

Let $P_{\mathbb{C}}$ be the saturated set $\mathbb{T}_{\mathbb{C}}^d.P$ of P . It is an open set of M . The next step is to consider the quotient of $P_{\mathbb{C}}$ by $\mathbb{T}_{\mathbb{C}}^d$. This has to be done carefully because the $\mathbb{T}_{\mathbb{C}}^d$ -action on $P_{\mathbb{C}}$ is not proper. Actually, the map

$$(10) \quad \mathfrak{t}_d \times P \rightarrow P_{\mathbb{C}}, \quad t, y \rightarrow l_{it}.y$$

is a diffeomorphism. So every $\mathbb{T}_{\mathbb{C}}^d$ -orbit of $P_{\mathbb{C}}$ intersects P in a \mathbb{T}^d -orbit and the injection $P \rightarrow P_{\mathbb{C}}$ induces a bijection from M_r onto $P_{\mathbb{C}}/\mathbb{T}_{\mathbb{C}}^d$. Furthermore every slice $U \subset P$ for the \mathbb{T}^d -action on P is a slice for the $\mathbb{T}_{\mathbb{C}}^d$ -action. Viewed as a quotient by a holomorphic action, the orbifold M_r inherits a complex structure. This complex structure is compatible with ω_r .

Similarly, the bundle L_r^k may be considered as the quotient of $L^k|_{P_{\mathbb{C}}}$ by the complex action and inherits a holomorphic structure. This is the unique holomorphic structure compatible with the connection and the Hermitian product. Denote by $p_{\mathbb{C}}$ the projection $P_{\mathbb{C}} \rightarrow M_r$ and observe that $p_{\mathbb{C}}^* L_r^k$ is naturally isomorphic with $L^k|_{P_{\mathbb{C}}}$.

The interest of viewing L_r^k and M_r as complex quotients is that there is a natural identification of the $\mathbb{T}_{\mathbb{C}}^d$ -invariant holomorphic sections Ψ of $L^k \rightarrow P_{\mathbb{C}}$ with the holomorphic sections Ψ_r of $L_r^k \rightarrow M_r$, given by $p_{\mathbb{C}}^* \Psi_r = \Psi$. So the map (8) takes its values in the space $\mathcal{H}_{r,k}$ of holomorphic sections of L_r^k .

Definition 3.7. $V_k : \mathcal{H}_{\lambda,k} \rightarrow \mathcal{H}_{r,k}$ is the map which sends Ψ into the section Ψ_r such that

$$p_{\mathbb{C}}^* \Psi_r = \left(\frac{2\pi}{k} \right)^{\frac{d}{4}} \Psi|_{P_{\mathbb{C}}} \quad \text{or equivalently} \quad p^* \Psi_r = \left(\frac{2\pi}{k} \right)^{\frac{d}{4}} j^* \Psi.$$

The rescaling by $(2\pi/k)^{\frac{d}{4}}$ is such that V_k and its inverse are bounded independently of k (cf. proposition 4.22).

Remark 3.8. (Orbifold). Let us detail the previous constructions when the \mathbb{T}^d -action is not free. For the basic definitions of the theory of orbifolds, our references are the section 14.1 of [10] and the appendix of [8].

As topological spaces, M_r and L_r^k are the quotients P/\mathbb{T}^d and j^*L^k/\mathbb{T}^d . M_r is naturally endowed with a collection of orbifold charts in the following way. Let $x \in P$, $G \subset \mathbb{T}^d$ be its isotropy subgroup and $U \subset P$ be a slice at x for the \mathbb{T}^d -action. Denote by π_U the projection $U \rightarrow M_r$ and by $|U| \subset M_r$ its image. Then $(|U|, U, G, \pi_U)$ is an orbifold chart of M_r , i.e. $|U|$ is an open set of M_r , U a manifold, G a finite group which acts on U by diffeomorphisms and π_U factors through a homeomorphism $U/G \rightarrow |U|$. These charts cover M_r and satisfy some compatibility conditions, which defines the orbifold structure of M_r .

For every such chart, the bundle L^k restricts to a G -bundle

$$L_{r,U}^k \rightarrow U.$$

These bundles are orbifold charts of the orbi-bundle $L_r^k \rightarrow M_r$. A section of L_r^k is a continuous section of $L_r^k \rightarrow M_r$ which lifts to a G -invariant C^∞ section of $L_{r,U}^k$ for every U . Since every \mathbb{T}^d -invariant section of L^k restricts to a G -invariant section of $L_{r,U}^k$, the map (8) is well-defined. Continuing in this way, we can introduce the Kähler structure of M_r , the Hermitian and holomorphic structures of L_r^k , its connection and verify that we obtain a well-defined map V_k as in definition 3.7.

It is also useful to consider P and $P_{\mathbb{C}}$ as orbifolds and the projections $p : P \rightarrow M_r$ and $p_{\mathbb{C}} : P_{\mathbb{C}} \rightarrow M_r$ as orbifold maps. For instance, let $(|U|, U, G, \pi_U)$ be a chart defined as above. Let

$$V := \mathbb{T}^d \times \mathfrak{t}_d \times U, \quad |V| := \mathbb{T}_{\mathbb{C}}^d \cdot U$$

and π_V be the map $V \rightarrow |V|$ which sends (θ, t, u) into $l_{\theta+it}.u$. Let G acts on V by

$$G \times V \rightarrow V, \quad g, (\theta, t, u) \rightarrow (\theta - g, t, l_g.u)$$

Then $(|V|, V, G, \pi_V)$ is an orbifold chart of $P_{\mathbb{C}}$. Furthermore $\mathbb{T}_{\mathbb{C}}^d$ acts on $V = \mathbb{T}_{\mathbb{C}}^d \times U$ by left multiplication, this action lifts the $\mathbb{T}_{\mathbb{C}}^d$ -action on $|V|$, and the projection $V \rightarrow U$ locally lifts $p_{\mathbb{C}}$:

$$(11) \quad \begin{array}{ccc} V & \longrightarrow & U \\ \downarrow \pi_V & & \downarrow \pi_U \\ |V| & \longrightarrow & |U| \end{array}$$

Now, instead of viewing U as a submanifold of P , we consider it as the quotient of V by $\mathbb{T}_{\mathbb{C}}^d$. To define the various structure on U , we can lift everything from $|V|$ to V and we perform the reduction from V to U . Since the $\mathbb{T}_{\mathbb{C}}^d$ -action on V is free, we are reduced to the manifold case. Furthermore since $V \rightarrow |V|$ is a G -principal bundle and $V \rightarrow U$ is G -equivariant, we obtain G -invariant structures. We can apply the same method with the map $p : P \rightarrow M_r$. \square

Remark 3.9. (proof of theorem 3.3). Since $P_{\mathbb{C}}$ is open, V_k is injective. That V_k is surjective is more difficult to prove. It consists to show that every invariant holomorphic section of $L^k \rightarrow P_{\mathbb{C}}$ extends to an invariant holomorphic section over

M . Let us precise that the proof of Guillemin and Sternberg extends to the orbifold case without modification. The only technical point is to show that there exists a non-vanishing section in $\mathcal{H}_{\lambda,k}$, when k is sufficiently large (theorem 5.6 of [13]). This will be proved in section 7.2, cf. remark 7.5. \square

Remark 3.10. (harmonic oscillator). The definition of $V_k : \mathcal{H}_{1,k} \rightarrow \mathcal{H}_{r,k}$ is the same. It is easily checked that this map is onto: every holomorphic section of L_r^k lifts to a holomorphic section Ψ of L^k over $\mathbb{C}^n - \{0\}$ satisfying

$$(12) \quad \Psi(u^{p_1} z_1, \dots, u^{p_n} z_n) = u^k \Psi(z_1, \dots, z_n).$$

Since it is bounded on a neighborhood of the origin, it extends on \mathbb{C}^n . Writing its Taylor expansion at the origin, we deduce from (12) that Ψ is polynomial and belongs to $\mathcal{H}_{1,k}$. \square

4. REDUCTION OF TOEPLITZ OPERATORS

4.1. Toeplitz operators. Let us denote by $L^2(M, L^k)$ the space of L^2 sections of L^k . We define the scalar product of sections of L^k as

$$(\Psi, \Psi')_M = \int_M (\Psi, \Psi')(x) \delta_M(x)$$

where (Ψ, Ψ') is the punctual scalar product and δ_M is the Liouville measure $\frac{1}{n!} |\omega^{\wedge n}|$. Let Π_k be the orthogonal projector of $L^2(M, L^k)$ onto \mathcal{H}_k .

Given $f \in C^\infty(M)$, we denote by M_f the operator of $L^2(M, L^k)$ sending Ψ into $f\Psi$. The set of symbols $S(M)$ consists of the sequences $(f(\cdot, k))_k$ of $C^\infty(M)$ which admit an asymptotic expansion of the form

$$(13) \quad f(\cdot, k) = \sum_{l=0}^{\infty} k^{-l} f_l + O(k^{-\infty}), \quad \text{with } f_0, f_1, \dots \in C^\infty(M),$$

for the C^∞ topology.

A Toeplitz operator is a family $(T_k)_k$ of the form

$$(14) \quad T_k = \Pi_k M_{f(\cdot, k)} \Pi_k + R_k$$

where $(f(\cdot, k)) \in S(M)$ and R_k is an operator of $L^2(M, L^k)$ satisfying $\Pi_k R_k \Pi_k = R_k$ and whose uniform norm is $O(k^{-\infty})$. The following result is a consequence of the works of Boutet de Monvel and Guillemin [4] (cf. [6]).

Theorem 4.1. *The set \mathcal{T} of Toeplitz operators is a $*$ -algebra. The contravariant symbol map*

$$\sigma_{\text{cont}} : \mathcal{T} \rightarrow C^\infty(M)[[\hbar]], \quad \Pi_k M_{f(\cdot, k)} \Pi_k + R_k \rightarrow \sum \hbar^l f_l$$

*is well-defined, onto and its kernel consists of the Toeplitz operators whose uniform norm is $O(k^{-\infty})$. Furthermore, the product $*_c$ induced on $C^\infty(M)[[\hbar]]$ is a star-product.*

The *principal* symbol of a Toeplitz operator is the first coefficient f_0 of its contravariant symbol. The operators $M_{\xi,k}$ are Toeplitz operators with principal symbol $\langle \mu, \xi \rangle$.

We use the same definitions and notations over M_r . Recall that λ is fixed and k runs over the positive multiples of κ . So a Toeplitz operator of \mathcal{T}_r is a family

$$(T_k)_{k=\kappa, 2\kappa, \dots}$$

We denote by $\Pi_{r,k}$ the orthogonal projector onto $\mathcal{H}_{r,k}$ and by $*_{cr}$ the product of the contravariant symbols of $C^\infty(M_r)[[\hbar]]$.

Remark 4.2. (Orbifold) In the case M_r is an orbifold, the definition of the Toeplitz operators makes sense. We will prove theorem 4.1 for the Toeplitz operators of M_r in section 6. \square

Remark 4.3. (harmonic oscillator) To avoid a discussion about the infinity of \mathbb{C}^n , we do not define the full algebra of Toeplitz operators on \mathbb{C}^n and do not state any result similar to theorem 4.1. We only consider the Toeplitz operators of the form

$$\Pi_k M_{f(.,k)} \Pi_k$$

where $f(.,k)$ is a symbol of $S(\mathbb{C}^n)$ (cf. definition in section 2.3). \square

4.2. Statement of the main result. Recall that V_k is the isomorphism from $\mathcal{H}_{\lambda,k}$ to $\mathcal{H}_{r,k}$ (cf. definition 3.7). Let U_k be the operator

$$L^2(M, L^k) \rightarrow L^2(M_r, L_r^k), \quad \Psi \rightarrow \begin{cases} V_k (V_k^* V_k)^{-\frac{1}{2}} \Psi, & \text{if } \Psi \in \mathcal{H}_{\lambda,k} \\ 0, & \text{if } \Psi \text{ is orthogonal to } \mathcal{H}_{\lambda,k} \end{cases}$$

Hence

$$U_k^* U_k = \Pi_{\lambda,k}, \quad U_k U_k^* = \Pi_{r,k}, \quad \Pi_{r,k} U_k \Pi_{\lambda,k} = U_k.$$

where $\Pi_{\lambda,k}$ is the orthogonal projector onto $\mathcal{H}_{\lambda,k}$. The main result of the section is the following theorem and the corresponding theorem 2.2 for the harmonic oscillator.

Theorem 4.4. *Let T_k be a Toeplitz operator of M with principal symbol f . Then*

$$U_k T_k U_k^* : L^2(M_r, L_r^k) \rightarrow L^2(M_r, L_r^k)$$

is a Toeplitz operator of M_r . Its principal symbol is the function $g \in C^\infty(M_r)$ such that

$$g(p(x)) = \int_{\mathbb{T}^d} f(l_\theta \cdot x) \delta_{\mathbb{T}^d}(\theta), \quad x \in P,$$

with $\delta_{\mathbb{T}^d}$ the Haar measure of \mathbb{T}^d .

In the following subsection we introduce the λ -Toeplitz operators. These are the operators of the form

$$\Pi_{\lambda,k} T_k \Pi_{\lambda,k} \quad \text{where } T_k \in \mathcal{T}.$$

In the next subsections, we prove some estimates for the sections of $\mathcal{H}_{\lambda,k}$ and introduce an integration map. Then we prove that the space of λ -Toeplitz operators is isomorphic to the space of Toeplitz operators of M_r , a stronger result than theorem 4.4.

Our proof uses the properties of the Toeplitz operators of M_r stated in theorem 4.1. So, in the case the \mathbb{T}^d -action is not free, the proof will be complete only in section 6 with the proof of theorem 4.1 for orbifolds.

For the following, we introduce the inverse $W_k : \mathcal{H}_{r,k} \rightarrow \mathcal{H}_{\lambda,k}$ of V_k . We consider that V_k and W_k act not only on $\mathcal{H}_{\lambda,k}$ and $\mathcal{H}_{r,k}$ respectively, but on the space of L^2 sections in such a way that they vanish on the orthogonal of $\mathcal{H}_{\lambda,k}$ and $\mathcal{H}_{r,k}$ respectively. So

$$\Pi_{r,k} V_k \Pi_{\lambda,k} = V_k, \quad \Pi_{\lambda,k} W_k \Pi_{r,k} = W_k, \quad V_k W_k = \Pi_{r,k}, \quad W_k V_k = \Pi_{\lambda,k},$$

and with the convention $0^{-\frac{1}{2}} = 0$, the equality $U_k = V_k (V_k^* V_k)^{-\frac{1}{2}}$ is valid on $L^2(M, L^k)$. Furthermore we say that a function or a section is invariant if it is invariant with respect to the action of \mathbb{T}^d .

4.3. The λ -Toeplitz operators. We begin with a useful formula for the orthogonal projector $\Pi_{\lambda,k}$ onto $\mathcal{H}_{\lambda,k}$. Denote by $P_{\lambda,k}$ the orthogonal projector of $L^2(M, L^k)$ onto the space of invariant sections of L^k (not necessarily holomorphic). If Ψ is a section of L^k , $P_{\lambda,k} \Psi$ is given by the well known formula

$$(15) \quad P_{\lambda,k} \Psi = \int_{\mathbb{T}^d} \mathcal{L}_\theta^* \Psi \delta_{\mathbb{T}^d}(\theta).$$

where $\delta_{\mathbb{T}^d}$ is the Haar measure. Since $P_{\lambda,k}$ sends \mathcal{H}_k in \mathcal{H}_k , we have

$$(16) \quad \Pi_{\lambda,k} = P_{\lambda,k} \Pi_k = \Pi_k P_{\lambda,k}.$$

Let \mathcal{T}_λ be the set of λ -Toeplitz operators

$$\mathcal{T}_\lambda := \{\Pi_{\lambda,k} T_k \Pi_{\lambda,k}; T_k \in \mathcal{T}\}.$$

This first result follows from theorem 4.1 about Toeplitz operators.

Theorem 4.5. \mathcal{T}_λ is a $*$ -algebra. Furthermore if T_k is a Toeplitz operator with contravariant symbol $\sum \hbar^l f_l$, then

$$(17) \quad \Pi_{\lambda,k} T_k \Pi_{\lambda,k} = \Pi_{\lambda,k} M_{f_\lambda(\cdot, k)} \Pi_{\lambda,k} + R_k$$

where R_k is $O(k^{-\infty})$, $\Pi_{\lambda,k} R_k \Pi_{\lambda,k} = R_k$ and $f_\lambda(\cdot, k)$ is an invariant symbol of M with an asymptotic expansion $\sum k^{-l} f_{\lambda,l}$ such that

$$f_{\lambda,l}(x) = \int_{\mathbb{T}^d} f_l(l_\theta \cdot x) \delta_{\mathbb{T}^d}(\theta), \quad l = 0, 1, \dots$$

So the λ -Toeplitz operators can also be defined as the operators of the form

$$(18) \quad \Pi_{\lambda,k} M_{f(\cdot, k)} \Pi_{\lambda,k} + R_k$$

where the multiplier $f(\cdot, k) \in S(M)$ is invariant, $\Pi_{\lambda,k} R_k \Pi_{\lambda,k} = R_k$ and R_k is $O(k^{-\infty})$.

A similar algebra was introduced by Guillemin and Sternberg [14] in the context of pseudodifferential operators. Their main theorem was that there exists an associated symbolic calculus, where the symbols are defined on the reduced space M_r . Let us state the corresponding result in our context.

Theorem 4.6. The map $\sigma_{\text{princ}} : \mathcal{T}_\lambda \rightarrow C^\infty(M_r)$, which associates to a λ -Toeplitz operator of the form (18) with an invariant multiplier $f(\cdot, k)$, the function $g_0 \in C^\infty(M_r)$ such that

$$f(\cdot, k) = p^* g_0 + O(k^{-1}) \quad \text{over } P$$

is well-defined. Furthermore the following sequence is exact

$$0 \longrightarrow \mathcal{T}_\lambda \cap O(k^{-1}) \longrightarrow \mathcal{T}_\lambda \xrightarrow{\sigma_{\text{princ}}} C^\infty(M_r) \longrightarrow 0.$$

Finally if T_k^1 and T_k^2 are λ -Toeplitz operators, then

$$\sigma_{\text{princ}}(T_k^1 \cdot T_k^2) = \sigma_{\text{princ}}(T_k^1) \cdot \sigma_{\text{princ}}(T_k^2)$$

So $[T_k^1, T_k^2]$ is $O(k^{-1})$ and $k[T_k^1, T_k^2]$ belongs to \mathcal{T}_λ . Its principal symbol is

$$\sigma_{\text{princ}}(k[T_k^1, T_k^2]) = i\{\sigma_{\text{princ}}(T_k^1), \sigma_{\text{princ}}(T_k^2)\}$$

where $\{.,.\}$ is the Poisson bracket of $C^\infty(M_r)$.

This theorem doesn't follow from theorem 4.1. Actually it is a corollary of theorem 4.25, which says that the algebra of λ -Toeplitz operators and the algebra of Toeplitz operators of M_r are isomorphic (cf. remark 4.27).

Proof of theorem 4.5. First let us prove the second point. Let

$$T_k = \Pi_k M_{f(\cdot, k)} \Pi_k + R_k$$

be a Toeplitz operator. Since $\Pi_k \Pi_{\lambda, k} = \Pi_{\lambda, k} \Pi_k = \Pi_{\lambda, k}$, we have

$$\Pi_{\lambda, k} T_k \Pi_{\lambda, k} = \Pi_{\lambda, k} M_{f(\cdot, k)} \Pi_{\lambda, k} + \Pi_{\lambda, k} R_k \Pi_{\lambda, k}.$$

Clearly R_k is $O(k^{-\infty})$ implies that $\Pi_{\lambda, k} R_k \Pi_{\lambda, k}$ is $O(k^{-\infty})$. Since $\Pi_{\lambda, k} P_{\lambda, k} = P_{\lambda, k} \Pi_{\lambda, k} = \Pi_{\lambda, k}$, we have

$$\Pi_{\lambda, k} M_{f(\cdot, k)} \Pi_{\lambda, k} = \Pi_{\lambda, k} P_{\lambda, k} M_{f(\cdot, k)} P_{\lambda, k} \Pi_{\lambda, k}.$$

Then using (15), we obtain

$$P_{\lambda, k} M_{f(\cdot, k)} P_{\lambda, k} = M_{f_\lambda(\cdot, k)} P_{\lambda, k}$$

where $f_\lambda(\cdot, k)$ is the invariant symbol

$$f_\lambda(x, k) = \int_{\mathbb{T}^d} f(l_\theta.x, k) \delta_{\mathbb{T}^d}(\theta).$$

Consequently,

$$\Pi_{\lambda, k} T_k \Pi_{\lambda, k} = \Pi_{\lambda, k} M_{f_\lambda(\cdot, k)} \Pi_{\lambda, k} + \Pi_{\lambda, k} R_k \Pi_{\lambda, k},$$

which gives the result. To prove that T_λ is a $*$ -algebra, the only difficulty is to check that the product of two λ -Toeplitz operators is a λ -Toeplitz operator. Let $f^1(\cdot, k)$ and $f^2(\cdot, k)$ be invariant symbols of $S(M)$. We have to show that

$$\Pi_{\lambda, k} M_{f^1(\cdot, k)} \Pi_{\lambda, k} M_{f^2(\cdot, k)} \Pi_{\lambda, k}$$

is a λ -Toeplitz operator. By (16),

$$\Pi_{\lambda, k} M_{f^1(\cdot, k)} \Pi_{\lambda, k} M_{f^2(\cdot, k)} \Pi_{\lambda, k} = \Pi_{\lambda, k} M_{f^1(\cdot, k)} \Pi_k P_{\lambda, k} M_{f^2(\cdot, k)} \Pi_{\lambda, k}$$

Since $f^2(\cdot, k)$ is invariant, $P_{\lambda, k}$ and $M_{f^2(\cdot, k)}$ commute, so

$$\begin{aligned} &= \Pi_{\lambda, k} M_{f^1(\cdot, k)} \Pi_k M_{f^2(\cdot, k)} P_{\lambda, k} \Pi_{\lambda, k} \\ &= \Pi_{\lambda, k} \Pi_k M_{f^1(\cdot, k)} \Pi_k M_{f^2(\cdot, k)} \Pi_k \Pi_{\lambda, k} \end{aligned}$$

by (16). Finally $\Pi_k M_{f^1(\cdot, k)} \Pi_k M_{f^2(\cdot, k)} \Pi_k$ is a Toeplitz operator since it is the product of two Toeplitz operators. \square

Remark 4.7. (harmonic oscillator) We define the λ -Toeplitz operators as the operators of the form

$$\Pi_{1, k} M_{f(\cdot, k)} \Pi_{1, k} + R_k$$

where $f(\cdot, k)$ belongs to $S(\mathbb{C}^n)$, $\Pi_{1, k}$ is the orthogonal projector onto $\mathcal{H}_{1, k}$, R_k satisfies $\Pi_{1, k} R_k \Pi_{1, k} = R_k$ and its uniform norm is $O(k^{-\infty})$. Then theorems 4.5 and 4.6 remain true. The proof that the multiplier $f(\cdot, k)$ can be chosen invariant is the same. The fact that these operators form an algebra and the definition and properties of the principal symbol are consequences of theorem 4.25. \square

4.4. Norm of the invariant states. In this section, we estimate the norm of the eigensections of $\mathcal{H}_{\lambda, k}$. We begin with the estimation over $P_{\mathbb{C}}$. Using the diffeomorphism (10), we identify $P_{\mathbb{C}}$ with $\mathfrak{t}_d \times P$. Let ξ_i be a basis of \mathfrak{t}_d and denote by t_i the associated linear coordinates.

Proposition 4.8. For every $\mathbb{T}_{\mathbb{C}}^d$ -invariant section Ψ of $L^k \rightarrow P_{\mathbb{C}}$, we have

$$(19) \quad (\Psi, \Psi)(t, y) = e^{-k\varphi(t, y)} (\Psi, \Psi)(0, y), \quad \forall (t, y) \in \mathfrak{t}_d \times P$$

where φ is the C^∞ function on $\mathfrak{t}_d \times P$ solution of the equations

$$\varphi(0, y) = 0, \quad \partial_{t_i} \varphi(t, y) = 2(\lambda_i - \mu_i(t, y)), \quad \text{with } i = 1, \dots, d$$

and $\mu_i := \langle \mu, \xi_i \rangle$, $\lambda_i := \langle \lambda, \xi_i \rangle$ are the components of μ and λ .

Proof. By equation (9), $i\xi \in i\mathfrak{t}_d$ acts on the sections of L^k by

$$\nabla_{J\xi^\#} - k\langle\mu - \lambda, \xi\rangle.$$

So if Ψ is a $\mathbb{T}_\mathbb{C}^d$ -invariant section,

$$\nabla_{J\xi^\#} \Psi = k\langle\mu - \lambda, \xi\rangle \Psi.$$

which leads to

$$(20) \quad (J\xi^\#) \cdot (\Psi, \Psi) = 2k\langle\mu - \lambda, \xi\rangle (\Psi, \Psi)$$

and shows the proposition. \square

On the complementary set $P_\mathbb{C}^c$ of $P_\mathbb{C}$ in M , the situation is simpler.

Proposition 4.9. *Every $\mathbb{T}_\mathbb{C}^d$ -invariant section Ψ of $L^k \rightarrow M$ vanishes over $P_\mathbb{C}^c$.*

Proof. This is also a consequence of equations (20) (cf. theorem 5.4 of [13]). \square

The previous propositions were shown by Guillemin and Sternberg in [13]. Furthermore they noticed the following important fact.

Lemma 4.10. *Let g be the Kähler metric ($g(X, Y) = \omega(X, JY)$). Then*

$$\frac{1}{2} \partial_{t_i} \partial_{t_j} \varphi(t, y) = g(\xi_i^\#, \xi_j^\#)(t, y).$$

Proof. We have $J\xi_i^\# \cdot \langle\mu, \xi_j\rangle = \omega(J\xi_i^\#, \xi_j^\#) = -g(\xi_i^\#, \xi_j^\#)$. The result follows from proposition 4.8. \square

Then for every $y \in P$, the function $\varphi(\cdot, y)$ is strictly convex. It admits a global minimum at $t = 0$ and this minimum is $\varphi(0, y) = 0$. We obtain the following proposition.

Theorem 4.11. *Let $\epsilon > 0$, P_ϵ be the subset of $P_\mathbb{C}$*

$$P_\epsilon := \{(t, y) \in P_\mathbb{C}; |t| < \epsilon\}$$

and P_ϵ^c its complementary subset in M . There exists some positive constants $C(\epsilon)$, C, C' such that for every k and every $\Psi \in \mathcal{H}_{\lambda, k}$,

$$(\Psi, \Psi)(x) \leq Ck^n e^{-kC(\epsilon)} (\Psi, \Psi)_M, \quad \forall x \in P_\epsilon^c$$

$$\text{and} \quad (\Psi, \Psi)_{P_\epsilon^c} \leq C'k^n e^{-kC(\epsilon)} (\Psi, \Psi)_M.$$

where $(\Psi, \Psi)_{P_\epsilon^c} := \int_{P_\epsilon^c} (\Psi, \Psi) \delta_M$

Remark 4.12. This result shows that the eigenstates of $\mathcal{H}_{\lambda, k}$ are concentrated on P . Actually, since the $T_{\xi, k}$ are Toeplitz operators with principal symbol $\langle\mu, \xi\rangle$, we could directly deduce from general properties of these operators [7] a weaker version where $Ck^n e^{-kC(\epsilon)}$ is replaced by $C_N(\epsilon)k^{-N}$ with N arbitrary large. \square

Proof. Let $C(\epsilon)$ be the minimum value of φ over the compact set

$$\{(t, y) \in P_\mathbb{C}; |t| = \epsilon\}.$$

Since $\varphi(\cdot, y)$ is strictly convex with a global vanishing minimum at $t = 0$, $C(\epsilon)$ is positive and

$$\varphi(t, y) \geq C(\epsilon), \quad \forall (y, t) \in P_\epsilon^c.$$

On the other hand, using coherent states as in section 5 of [6], we prove that there exists a constant C such that for every k and $\Psi \in \mathcal{H}_k$,

$$(\Psi, \Psi)(x) \leq Ck^n (\Psi, \Psi)_M, \quad \forall x \in P.$$

If Ψ belongs to $\mathcal{H}_{\lambda,k}$, then it is $\mathbb{T}_{\mathbb{C}}^d$ -invariant and so it satisfies equation (19). Furthermore it vanishes over the complementary set of $P_{\mathbb{C}}$. This implies the first part of the result. By integrating, we get the second part with $C' = C \text{Vol}(P_{\epsilon})$. \square

Remark 4.13. (harmonic oscillator) We have to adapt theorem 4.11 since \mathbb{C}^n is not compact. The imaginary part of the complex action is given by

$$\mathbb{R} \times \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad (t, z) \rightarrow l_{it}.z = (e^{-t\mathbf{p}_1} z_1, \dots, e^{-t\mathbf{p}_n} z_n).$$

A section Ψ of $\mathcal{H}_{1,k}$ satisfies

$$(21) \quad \Psi(l_{it}.z) = e^{-kt} \Psi(z)$$

From this we obtain the following lemma.

Lemma 4.14. *Let $g(.,k)$ be a sequence of functions on \mathbb{C}^n . Assume that there exists C and N such that*

$$|g(z,k)| \leq C(1+|z|)^N, \quad \forall z \in \mathbb{C}^n, \forall k.$$

Let $\epsilon > 0$, P_{ϵ} be the subset $\{l_{it}.z; |t| < \epsilon \text{ and } z \in P\}$ of $P_{\mathbb{C}}$ and P_{ϵ}^c its complementary subset in \mathbb{C}^n . There exists some positive constants $C(\epsilon)$, C' such that for every k and every $\Psi \in \mathcal{H}_{1,k}$,

$$(g(.,k)\Psi, \Psi)_{P_{\epsilon}^c} \leq C' k^n e^{-kC(\epsilon)} (\Psi, \Psi)_{\mathbb{C}^n}.$$

Proof. Let Ψ belongs to $\mathcal{H}_{1,k}$. First, as in the compact case, there exists C_1 such that

$$|\Psi(z)|^2 e^{-k|z|^2} \leq C_1 k^n (\Psi, \Psi)_{\mathbb{C}^n}, \quad \forall z \in P$$

Let $w = l_{it}.z$ with $z \in P$. It follows from (21) that

$$\begin{aligned} |\Psi(w)|^2 e^{-k|w|^2} &= |\Psi(z)|^2 e^{-k|z|^2} e^{-k(|w|^2 + 2t - |z|^2)} \\ &\leq C_1 k^n e^{-k(|w|^2 + 2t - |z|^2)} (\Psi, \Psi)_{\mathbb{C}^n} \end{aligned}$$

Using that $|z|^2 \leq 1$, we obtain

$$(22) \quad |\Psi(w)|^2 e^{-k|w|^2} \leq C_1 k^n e^{-k} (\Psi, \Psi)_{\mathbb{C}^n} \quad \text{if } t \geq 1.$$

On the other hand, assume that $t < 0$. There exists $C_2 > 0$ such that $|z|^2 \geq C_2$ for every $z \in P$. So $|w|^2 = \sum e^{-2\mathbf{p}_i t} |z_i|^2 \geq C_2 e^{-2t}$. Consequently

$$\begin{aligned} e^{-k(|w|^2 + 2t - |z|^2)} &\leq C_3 e^{-k(|w|^2 - 2 \ln |w| - 1)} \\ &\leq C_3 e^{-\frac{k}{2}|w|^2} e^{-k}, \quad \text{if } |w| \text{ is sufficiently large.} \end{aligned}$$

Hence there exists $t_- < 0$ such that

$$|\Psi(w)|^2 e^{-k|w|^2} \leq C_4 k^n e^{-k} (\Psi, \Psi)_{\mathbb{C}^n} e^{-\frac{k}{2}|w|^2} \quad \text{if } t \leq t_-.$$

This last equation and (22) lead to the result if $\epsilon \geq \max(1, -t_-)$. For the smaller values of ϵ , we complete the proof as in theorem 4.11 since we are reduced to a compact subset of $P_{\mathbb{C}}$. \square

4.5. The integration map. Recall that δ_M and δ_{M_r} are the Liouville measures of M and M_r respectively and $p_{\mathbb{C}}$ denote the projection $P_{\mathbb{C}} \rightarrow M_r$.

Let $I_k : C_o^{\infty}(P_{\mathbb{C}}) \rightarrow C^{\infty}(M_r)$ be the map given by

$$I_k(f)(x) = \left(\frac{k}{2\pi}\right)^{\frac{d}{2}} \left(\int_{p_{\mathbb{C}}^{-1}(x)} e^{-k\varphi} f \delta_M \right) \cdot \delta_{M_r}^{-1}(x)$$

where φ is defined in proposition 4.8. Equivalently, $I_k(f) \delta_{M_r}$ is the push-forward of $(k/2\pi)^{\frac{d}{2}} e^{-k\varphi} f \delta_M$ by $p_{\mathbb{C}}$.

Remark 4.15. (Orbifold). If M_r is an orbifold, the push-forward by $p_{\mathbb{C}}$ of a density ν of $P_{\mathbb{C}}$ with compact support is defined as in the manifold case in such a way that:

$$\forall g \in C^\infty(M_r), \int_{P_{\mathbb{C}}} \nu p_{\mathbb{C}}^* g = \int_{M_r} g p_{\mathbb{C}*} \nu.$$

Applying this in orbifold charts of $P_{\mathbb{C}}$ and M_r , we recover the usual definition. With the same notations as in remark 3.8, assume that g has a compact support in $|U|$. Denote by g_U , $(p_{\mathbb{C}*} \nu)_U$ the local lifts in U of g and $p_{\mathbb{C}*} \nu$. Then by definition of an integral in an orbifold,

$$\int_{M_r} g p_{\mathbb{C}*} \nu = \frac{1}{\#G} \int_U g_U \cdot (p_{\mathbb{C}*} \nu)_U$$

So it follows from (11) that $(p_{\mathbb{C}*} \nu)_U$ is the push-forward of $\pi_V^* \nu$ by the projection $V \rightarrow U$. One can check that the $(p_{\mathbb{C}*} \nu)_U$ agree on overlaps and define a global section $p_{\mathbb{C}*} \nu$. \square

Remark 4.16. (harmonic oscillator). Since we apply I_k only to functions with compact support, all the results in this section extend directly to this case. \square

We introduced the map I_k because it satisfies the following property.

Proposition 4.17. *For every $f \in C_o^\infty(P_{\mathbb{C}})$, we have*

$$(23) \quad (f\Psi, \Psi')_M = (I_k(f) V_k \Psi, V_k \Psi')_{M_r}, \quad \forall \Psi, \Psi' \in \mathcal{H}_{\lambda, k}$$

or equivalently

$$\Pi_{\lambda, k} M_f \Pi_{\lambda, k} = V_k^* M_{I_k(f)} V_k.$$

Proof. Let us prove (23). Since the support of f is a subset of $P_{\mathbb{C}}$,

$$(f\Psi, \Psi')_M = \int_{P_{\mathbb{C}}} f(\Psi, \Psi') \delta_M$$

By definition of V_k , we have

$$p^*(V_k \Psi, V_k \Psi') = \left(\frac{2\pi}{k}\right)^{\frac{d}{2}} j^*(\Psi, \Psi').$$

So equation (19) can be rewritten as

$$(\Psi, \Psi') = \left(\frac{k}{2\pi}\right)^{\frac{d}{2}} e^{-k\varphi} p_{\mathbb{C}}^*(V_k \Psi, V_k \Psi').$$

Hence,

$$\begin{aligned} (f\Psi, \Psi')_M &= \left(\frac{k}{2\pi}\right)^{\frac{d}{2}} \int_{P_{\mathbb{C}}} f e^{-k\varphi} p_{\mathbb{C}}^*(V_k \Psi, V_k \Psi') \\ &= \int_{M_r} I_k(f) (V_k \Psi, V_k \Psi') \delta_{M_r} \\ &= (I_k(f) V_k \Psi, V_k \Psi')_{M_r}. \end{aligned}$$

which proves the result. \square

For the following, we need to control the asymptotic behavior of $I_k(f)$ as k tends to infinity when f depends also on k . First observe that there is no restriction to consider only invariant functions of $C_o^\infty(P_{\mathbb{C}})$. Indeed, if $f \in C_o^\infty(P_{\mathbb{C}})$ and

$$f_\lambda(x) = \int_{\mathbb{T}^d} f(l_\theta \cdot x) \delta_{\mathbb{T}^d}(\theta)$$

then $I_k(f) = I_k(f_\lambda)$.

Denote by $S_o(P_{\mathbb{C}})$ the set of sequences $(f(., k))_k$ of $C_{\mathbb{T}^d}^\infty(P_{\mathbb{C}})$ such that there exists a compact set $K \subset P_{\mathbb{C}}$ which contains the support of $f(., k)$ for every k and $f(., k)$ admits an asymptotic expansion for the C^∞ topology of the form

$$f(., k) = \sum_l k^{-l} f_l + O(k^{-\infty}).$$

Let us introduce a basis ξ_i of the integral lattice of \mathfrak{t}_d and denote by $\xi_i^\#$ the associated vector fields of M . Recall that g is the Kähler metric. The following result involves the determinant of $g(\xi_i^\#, \xi_j^\#)$, which clearly doesn't depend on the choice of the basis ξ_i .

Proposition 4.18. *Let $f \in S_o(P_{\mathbb{C}})$. Then the sequence $I_k(f(., k))$ is a symbol of M_r . Furthermore*

$$p^* g_0 = j^* (\det[g(\xi_i^\#, \xi_j^\#)]^{\frac{1}{2}} \cdot f_0)$$

where g_0 and f_0 are such that $I_k(f(., k)) = g_0 + O(k^{-1})$ and $f(., k) = f_0 + O(k^{-1})$.

This proposition admits the following converse.

Corollary 4.19. *For every symbol $g(., k)$ of $S(M_r)$, there exists $f(., k) \in S_o(P_{\mathbb{C}})$ such that $g(., k) = I_k(f(., k))$.*

Proof of corollary 4.19. Let r be a non negative invariant function of $C_o^\infty(P_{\mathbb{C}})$ such that $r = 1$ on a neighborhood of P . Set

$$f(., k) := r \cdot p_{\mathbb{C}}^* (g(., k) I_k^{-1}(r)).$$

From the previous proposition, $I_k(r)$ is a symbol of M_r and the first coefficient of its asymptotic expansion doesn't vanish. So $I_k^{-1}(r)$ is also a symbol of M_r . Consequently $f(., k)$ belongs to $S_o(P_{\mathbb{C}})$. Furthermore, we have $g(., k) = I_k(f(., k))$. \square

Proof of proposition 4.18. We integrate first over the fibers of $\mathfrak{t}_d \times P \rightarrow P$. Let us compute the Liouville measure δ_M . We denote by t_1, \dots, t_d the linear coordinates of \mathfrak{t}_d associated to ξ_1, \dots, ξ_d .

Lemma 4.20. *There exists an invariant measure δ_P on P and a function $\delta \in C^\infty(\mathfrak{t}_d \times P)$ such that*

$$\begin{aligned} \delta_M &= \delta \cdot \delta_P \cdot |dt_1 \dots dt_d| \text{ over } \mathfrak{t}_d \times P, \\ p_* \delta_P &= \delta_{M_r} \quad \text{and} \quad \delta(0, .) = j^* \det[g(\xi_i^\#, \xi_j^\#)]. \end{aligned}$$

Proof. Let us write over $\{0\} \times P \subset \mathfrak{t}_d \times P$

$$\omega = \beta + \sum_{1 \leq i \leq d} \beta_i \wedge dt_i + \sum_{1 \leq i < j \leq d} a_{ij} dt_i \wedge dt_j$$

where $\beta \in \Omega^2(P)$, $\beta_i \in \Omega^1(P)$ and $a_{ij} \in C^\infty(P)$. Since this decomposition is unique, these forms are all invariant. Let us set

$$\delta_P = \frac{|\beta^{\wedge n_r}|}{n_r!} \cdot \frac{|\beta_1 \wedge \dots \wedge \beta_d|}{\det[g(\xi_i^\#, \xi_j^\#)]}$$

Since $j^* \omega = \beta$, $\beta = p^* \omega_r$. We also have $g(\xi_i^\#, \xi_j^\#) = \omega(\xi_i^\#, J\xi_j^\#) = \langle \beta_j, \xi_i^\# \rangle$. Since the ξ_i are a basis of the integer lattice, we obtain $p_* \delta_P = \delta_{M_r}$. In the case M_r is an orbifold, this can be proved using local charts of M_r and P as in remark 4.15.

Since $\beta(\xi_i^\#, .) = 0$, $(dt_i \wedge dt_j)(\xi_k^\#, .) = 0$ and $(\beta_i \wedge dt_i)(\xi_k^\#, \xi_l^\#) = 0$, we have over $\{0\} \times P$

$$\omega^{\wedge n} = \frac{n!}{n_r!} \beta^{\wedge n_r} \wedge (\beta_1 \wedge dt_1) \wedge \dots \wedge (\beta_d \wedge dt_d)$$

Hence

$$\delta_M = \det[g(\xi_i^\#, \xi_j^\#)] \delta_P \cdot |dt_1 \dots dt_d|$$

over $\{0\} \times P$, which proves the result. \square

Let $J_k(f)$ be the function of $C^\infty(P)$

$$J_k(f)(y) = \left(\frac{k}{2\pi}\right)^{\frac{d}{2}} \int_{t_d} e^{-k\varphi(t,y)} f(t,y) \delta(t,y) |dt_1 \dots dt_d|$$

It is invariant and $p^*I_k(f) = J_k(f)$. So we just have to estimate $J_k(f)(y)$ which can be done with the stationary phase lemma. Recall that we computed in lemma 4.10 the second derivatives of φ . The result follows. \square

Remark 4.21. The proof actually gives more about the map

$$(24) \quad F : C_{\mathbb{T}^d}^\infty(M)[[\hbar]] \rightarrow C^\infty(M_r)[[\hbar]], \quad \sum \hbar^l f_l \rightarrow \sum \hbar^l g_l$$

such that $I_k(f(\cdot, k)) = \sum k^{-l} g_l + O(k^{-\infty})$ if $f(\cdot, k) = \sum k^{-l} f_l + O(k^{-\infty})$.

Since F enters in the computation of the contravariant symbol of the reduced operator, let us give its properties. First it is $\mathbb{C}[[\hbar]]$ -linear. So

$$F = \sum \hbar^l F_l \quad \text{with } F_l : C_{\mathbb{T}^d}^\infty(M) \rightarrow C^\infty(M_r).$$

The operators F_l are of the following form

$$p^* F_l(g) = \det^{\frac{1}{2}}[g(\xi_i^\#, \xi_j^\#)] \sum_{|\alpha| \leq 2l} a_{\alpha,l} j^* \left((J\xi_1^\#)^{\alpha(1)} \dots (J\xi_d^\#)^{\alpha(d)} \cdot f \right)$$

where the functions $a_{l,\alpha}$ are polynomials in the derivatives of $\mu_i = \langle \mu, \xi_i \rangle$ and $\Delta\mu_i$ with respect to the gradient vector fields of the μ_j .

Indeed by proposition 4.8 the derivatives of φ can be computed in terms of the derivatives of μ_i . Furthermore it is easily proved that

$$(J\xi_i^\#) \cdot \ln \delta = \Delta\mu_i$$

with Δ the Laplace-Beltrami operator of M , which gives the derivatives of δ in terms of the derivatives of $\Delta\mu_i$. Then the computation of the functions $a_{\alpha,l}$ follows from the stationary phase lemma. \square

4.6. From the λ -Toeplitz operators to the reduced Toeplitz operators. We begin with a rough estimate of the maps V_k and W_k .

Proposition 4.22. *There exists a constant $C > 0$, such that for every k the uniform norms of V_k , V_k^* , W_k and W_k^* are bounded by C .*

Proof. By corollary 4.19, there exists $f(\cdot, k) \in S_o(P_C)$ such that $I_k(f(\cdot, k)) = 1$. By proposition 4.17,

$$V_k^* V_k = \Pi_{\lambda,k} M_{f(\cdot, k)} \Pi_{\lambda,k}.$$

Furthermore it follows from proposition 4.18 that $f(\cdot, k) = f_0 + O(k^{-1})$ with f_0 positive on P .

Since $f(\cdot, k)$ is a symbol, there exists C_1 such that $f(\cdot, k) \leq C_1$ over M for every k . So

$$(V_k \Psi, V_k \Psi)_{M_r} = (f(\cdot, k) \Psi, \Psi)_M \leq C_1 (\Psi, \Psi)_M$$

which proves that the uniform norms of V_k and V_k^* are smaller than $C_1^{\frac{1}{2}}$.

Since f_0 is positive on P , there exists a neighborhood P_ϵ of P defined as in theorem 4.11 and a constant $C_2 > 0$, such that

$$f(\cdot, k) \geq C_2, \quad \text{over } P_\epsilon$$

when k is sufficiently large. So

$$(V_k \Psi, V_k \Psi)_{M_r} = (f(\cdot, k) \Psi, \Psi)_M \geq (f(\cdot, k) \Psi, \Psi)_{P_\epsilon} \geq C_2(\Psi, \Psi)_{P_\epsilon}$$

Furthermore theorem 4.11 implies that

$$(\Psi, \Psi)_{P_\epsilon} = (\Psi, \Psi)_M - (\Psi, \Psi)_{P_\epsilon^c} \geq \frac{1}{2}(\Psi, \Psi)_M$$

when k is sufficiently large. Consequently the uniform norms of W_k and W_k^* are smaller than $(\frac{1}{2}C_2)^{-\frac{1}{2}}$ when k is sufficiently large. \square

Remark 4.23. (harmonic oscillator) The result is still valid. There are some modifications in the proof. Instead of theorem 4.11, we have to use lemma 4.14. The same holds for theorems 4.24 and 4.25. \square

Let us now give the relations between the λ -Toeplitz operators and the Toeplitz operators of M_r .

Theorem 4.24. *If T_k is a λ -Toeplitz operator, then $W_k^* T_k W_k$ is a Toeplitz operator of M_r . Furthermore, if*

$$T_k = \Pi_{\lambda, k} M_{f(\cdot, k)} \Pi_{\lambda, k} + O(k^{-\infty}),$$

with $f(\cdot, k) = \sum k^{-l} f_l + O(k^\infty)$ an invariant symbol, then

$$\sigma_{\text{cont}}(W_k^* T_k W_k) = F(\sum h^l f_l)$$

where the map F is defined in (24). Conversely, if T_k is a Toeplitz operator of M_r , then $V_k^ T_k V_k$ is a λ -Toeplitz operator. So the map*

$$\mathcal{T}_\lambda \rightarrow \mathcal{T}_r, \quad T_k \rightarrow W_k^* T_k W_k$$

is a bijection.

Proof. Let

$$T_k = \Pi_{\lambda, k} M_{f(\cdot, k)} \Pi_{\lambda, k} + R_k$$

be a λ -Toeplitz operator, where $f(\cdot, k) = \sum k^{-l} f_l + O(k^\infty)$ is an invariant symbol and R_k is $O(k^{-\infty})$. Then

$$W_k^* T_k W_k = W_k^* M_{f(\cdot, k)} W_k + W_k^* R_k W_k$$

By proposition 4.22, $W_k^* R_k W_k$ is $O(k^{-\infty})$.

Let P_ϵ be a neighborhood of P defined as in theorem 4.11. Let r be an invariant function of $C_\infty^\infty(P_\mathbb{C})$ such that $r = 1$ over P_ϵ . Write

$$W_k^* M_{f(\cdot, k)} W_k = W_k^* M_{rf(\cdot, k)} W_k + W_k^* M_{(1-r)f(\cdot, k)} W_k$$

The second term on the right side is $O(k^{-\infty})$. Indeed by proposition 4.22, it suffices to prove that $\Pi_{\lambda, k} M_{(1-r)f(\cdot, k)} \Pi_{\lambda, k}$ is $O(k^{-\infty})$. We have

$$((1-r)f(\cdot, k) \Psi, (1-r)f(\cdot, k) \Psi)_M = ((1-r)f(\cdot, k) \Psi, (1-r)f(\cdot, k) \Psi)_{P_\epsilon^c}$$

since $r = 1$ over P_ϵ

$$\leq C(\Psi, \Psi)_{P_\epsilon^c}$$

where C doesn't depend of k . Theorem 4.11 leads to the conclusion.

Now $rf(\cdot, k)$ is a symbol of $S_o(P_\mathbb{C})$. So by proposition 4.18, $g(\cdot, k) := I_k(rf(\cdot, k))$ is a symbol of $S(M_r)$. By proposition 4.17,

$$W_k^* M_{rf(\cdot, k)} W_k = \Pi_{r, k} M_{g(\cdot, k)} \Pi_{r, k}.$$

which proves that $W_k^* T_k W_k$ is a Toeplitz operator of M_r with contravariant symbol $F(\sum h^l f_l)$.

Conversely, let

$$T_k = \Pi_{r,k} M_{g(\cdot,k)} \Pi_{r,k} + R_k$$

be a Toeplitz operator of M_r , where $g(\cdot, k) \in S(M_r)$ and R_k is $O(k^{-\infty})$. Write

$$V_k^* T_k V_k = V_k^* M_{g(\cdot,k)} V_k + V_k^* R_k V_k.$$

Then, $V_k^* R_k V_k$ is $O(k^{-\infty})$ by proposition 4.22. By corollary 4.19, there exists $f(\cdot, k) \in S_o(P_{\mathbb{C}})$ such that

$$V_k^* M_{g(\cdot,k)} V_k = \Pi_{\lambda,k} M_{f(\cdot,k)} \Pi_{\lambda,k}.$$

Consequently $V_k^* T_k V_k$ is a λ -Toeplitz operator. \square

Recall that $U_k = V_k (V_k^* V_k)^{-\frac{1}{2}} : L^2(M, L^k) \rightarrow L^2(M_r, L_r^k)$. Let us state our main result.

Theorem 4.25. *The map*

$$\mathcal{T}_{\lambda} \rightarrow \mathcal{T}_r, \quad T_k \rightarrow U_k T_k U_k^*$$

is an isomorphism of $$ -algebra. Furthermore, if*

$$T_k = \Pi_{\lambda,k} M_{f(\cdot,k)} \Pi_{\lambda,k} + O(k^{-\infty}),$$

with $f(\cdot, k) = \sum k^{-l} f_l + O(k^{\infty})$ an invariant symbol, then

$$\sigma_{\text{cont}}(U_k T_k U_k^*) = e^{-\frac{1}{2}} *_{\text{cr}} F(\sum \hbar^l f_l) *_{\text{cr}} e^{-\frac{1}{2}}$$

*where $e = F(1)$ and $e^{-\frac{1}{2}}$ is the formal series of $C^{\infty}(M_r)[[\hbar]]$ whose first coefficient is positive and such that $e^{-\frac{1}{2}} *_{\text{cr}} e^{-\frac{1}{2}} *_{\text{cr}} e = 1$.*

Again, in the proof we use some basic properties of the Toeplitz operators of M_r , which are known in the manifold case and will be extended to the orbifold case in section 6.

Proof. It is easily checked that

$$U_k = (W_k^* W_k)^{-\frac{1}{2}} W_k^*.$$

Let T_k be a λ -Toeplitz operator. We have

$$(25) \quad U_k T_k U_k^* = (W_k^* W_k)^{-\frac{1}{2}} W_k^* T_k W_k (W_k^* W_k)^{-\frac{1}{2}}$$

By theorem 4.24, $W_k^* W_k$ is a Toeplitz operator of M_r with a positive principal symbol. It follows from the functional calculus for Toeplitz operator (cf. [6]) that $(W_k^* W_k)^{-\frac{1}{2}}$ is a Toeplitz operator also. Now $W_k^* T_k W_k$ is a Toeplitz operator by theorem 4.24. Since the Toeplitz operators of M_r form an algebra, $U_k T_k U_k^*$ is a Toeplitz operator. The computation of its covariant symbol is also a consequence of (25). Indeed by theorem 4.24, the symbol of $W_k^* T_k W_k$ and $W_k^* W_k$ are $F(\sum \hbar^l f_l)$ and e respectively.

Conversely, if S_k is a Toeplitz operator of M_r , then

$$\begin{aligned} U_k^* S_k U_k &= W_k (W_k^* W_k)^{-\frac{1}{2}} S_k (W_k^* W_k)^{-\frac{1}{2}} W_k^* \\ &= (V_k^* W_k^*) W_k (W_k^* W_k)^{-\frac{1}{2}} S_k (W_k^* W_k)^{-\frac{1}{2}} W_k^* (W_k V_k) \\ &= V_k^* (W_k^* W_k)^{\frac{1}{2}} S_k (W_k^* W_k)^{\frac{1}{2}} V_k. \end{aligned}$$

And in a similar way, we deduce from theorem 4.24 that $U_k^* S_k U_k$ is a λ -Toeplitz operator. \square

Remark 4.26. (symbolic calculus). Recall that we computed the operator F at the end of section 4.5. Furthermore the star-product $*_{\text{cr}}$ can be computed in terms of the Kähler metric of M_r (cf. [6]). This leads to the computation of the contravariant symbol of the reduced operator $U_k T_k U_k^*$ in terms of the multiplier $\sum k^{-l} f_l$ defining the λ -Toeplitz operator T_k . In particular, the principal symbol g_0 of $U_k T_k U_k^*$ is such that $p^* g_0 = i^* f_0$. \square

Remark 4.27. (proof of theorem 4.6). Because of the previous remark, the λ -Toeplitz operator T_k and the Toeplitz operator $U_k T_k U_k^*$ have the same principal symbol. Consequently all the assertions of theorem 4.6 follow from theorem 4.25 and the calculus of the contravariant symbol for the Toeplitz operators of M_r . \square

Remark 4.28. (proof of theorem 4.4). Theorem 4.4 stated in the introduction is a consequence of theorems 4.5 and 4.25. To compute the contravariant symbol of the reduced operator, we have first to average the contravariant symbol of the Toeplitz operator of M and then apply the formula of theorem 4.25. \square

Remark 4.29. (V_k is not unitary). The fact that the Guillemin-Sternberg isomorphism is not unitary, even after the rescaling with the factor $(\frac{k}{2\pi})^{\frac{d}{4}}$, can be deduced from the spectral properties of the Toeplitz operators. Indeed by theorem 4.24, $W_k^* W_k$ is a Toeplitz operator with principal symbol g_0 such that

$$p^* g_0 = j^* \det[g(\xi_i^\#, \xi_j^\#)]^{\frac{1}{2}}.$$

Denote by m and M the minimum and maximum of g_0 . Then the smallest eigenvalue E_s of $W^* W$ and the biggest E_S are estimated by

$$E_s = m + O(k^{-1}), \quad E_S = M + O(k^{-1}).$$

So when the function $\det[g(\xi_i^\#, \xi_j^\#)]^{\frac{1}{2}}$ is not constant over P , W_k is not unitary when k is sufficiently large. This happens for instance in the case of the harmonic oscillator when the reduced space is not a manifold. \square

Remark 4.30. From a semi-classical point of view, the operator U_k is not unique. Indeed we can replace it with any operator of the form

$$T_k U_k S_k$$

with S_k a unitary λ -Toeplitz operator and T_k a unitary Toeplitz operator of M_r . We can state a theorem similar to theorem 4.25 with this operator. The only changes are in the symbolic calculus. \square

5. FOURIER INTEGRAL OPERATORS

In this section, we prove that the λ -Toeplitz operators, the Guillemin-Sternberg isomorphism and its unitarization are Fourier integral operators in the sense of [7]. Using this we can interpret the relations between these operators and the Toeplitz operators as compositions of Fourier integral operators corresponding to compositions of canonical relations.

We assume that the reduced space M_r is a manifold. A part of the material will be adapted to orbifolds in section 7.

5.1. Definitions. We first recall some definitions of [7]. Let M_1 and M_2 be compact Kähler manifolds endowed with prequantization bundles $L_1^\kappa \rightarrow M_1$ and $L_2^\kappa \rightarrow M_2$. Here κ is some fixed positive integer and in the following k is always a positive multiple of κ . Denote by \mathcal{H}_k^1 (resp. \mathcal{H}_k^2) the space of holomorphic sections of L_1^k (resp. L_2^k) and by Π_k^1 (resp. Π_k^2) the orthogonal projector onto \mathcal{H}_k^1 (resp. \mathcal{H}_k^2).

Consider a sequence $(T_k)_{k \in \kappa \mathbb{N}^*}$ such that for every k , T_k is an operator $\mathcal{H}_k^2 \rightarrow \mathcal{H}_k^1$. As previously we extend T_k to the Hilbert space of sections with finite norm in such

a way that it vanishes on the orthogonal of \mathcal{H}_k^2 . The Schwartz kernel T_k is the section of $L_1^k \boxtimes L_2^{-k} \rightarrow M_1 \times M_2$ such that

$$T_k \cdot \Psi(x_1) = \int_{M_2} T_k(x_1, x_2) \cdot \Psi(x_2) \delta_{M_2}(x_2)$$

where δ_{M_2} is the Liouville measure of M_2 . All the operators we consider in this section are of this form.

5.1.1. Smoothing operators. A sequence $(f(., k))$ of functions on a manifold X is $O_\infty(k^{-\infty})$ if for every compact set K , every $N \geq 0$, every vector fields Y_1, \dots, Y_N on X and every l , there exists C such that

$$|Y_1 \cdot Y_2 \dots Y_N f(., k)| \leq C k^{-l} \quad \text{on } K.$$

Let $L_X \rightarrow X$ be a Hermitian line bundle. Let (Ψ_k) be a sequence such that for every k , Ψ_k is a section of L_X^k . Then (Ψ_k) is $O_\infty(k^{-\infty})$ if for every local unitary section $t : V \rightarrow L_X$, the sequence $(f(., k))$ such that $\Psi_k = f(., k)t^k$ is $O_\infty(k^{-\infty})$.

We say that an operator (T_k) is smoothing if the sequence (T_k) of Schwartz kernels is $O_\infty(k^{-\infty})$. Clearly if T_k is $O_\infty(k^{-\infty})$, the compactness of $M_1 \times M_2$ implies that the uniform norm of T_k is $O(k^{-\infty})$. If $\Pi_k^1 T_k \Pi_k^2 = T_k$ for every k , then the converse is true.

5.1.2. Fourier integral operators. If ω_2 is the symplectic form of M_2 , we denote by M_2^- the manifold M_2 endowed with the symplectic form $-\omega_2$. The data to define a Fourier integral operator are a Lagrangian submanifold Γ of $M_1 \times M_2^-$, a flat unitary section t_Γ^κ of $L_1^\kappa \boxtimes L_2^{-\kappa} \rightarrow \Gamma$ and a formal series $\sum \hbar^l g_l$ of $C^\infty(\Gamma)[[\hbar]]$.

By definition T_k is a *Fourier integral operator* associated to $(\Gamma, t_\Gamma^\kappa)$ with *total symbol* $\sum \hbar^l g_l$ if on every compact set $K \subset M_1 \times M_2$ such that $K \cap \Gamma = \emptyset$,

$$T_k(x_1, x_2) = O_\infty(k^{-\infty})$$

Furthermore on a neighborhood U of Γ ,

$$(26) \quad T_k(x_1, x_2) = \left(\frac{k}{2\pi}\right)^{n(\Gamma)} E_\Gamma^\kappa(x_1, x_2) f(x_1, x_2, k) + O_\infty(k^{-\infty})$$

where

- i . E_Γ^κ is a section of $L_1^\kappa \boxtimes L_2^{-\kappa} \rightarrow U$ such that $E_\Gamma^\kappa = t_\Gamma^\kappa$ over Γ , and for every holomorphic vector field Z_1 of M_1 and Z_2 of M_2

$$\nabla_{(Z_1, 0)} E_\Gamma^\kappa \equiv 0 \text{ and } \nabla_{(0, \bar{Z}_2)} E_\Gamma^\kappa \equiv 0$$

modulo a section which vanishes to any order along Γ . Furthermore

$$|E_\Gamma^\kappa(x_1, x_2)| < 1$$

if $(x_1, x_2) \notin \Gamma$.

- ii . $(f(., k))_k$ is a symbol of $S(U)$ with an asymptotic expansion $\sum k^{-l} f_l$ such that

$$f_l = g_l \text{ over } \Gamma$$

and $(Z_1, 0) \cdot f_l \equiv 0$ and $(0, \bar{Z}_2) \cdot f_l \equiv 0$ modulo a function which vanishes to any order along Γ for every holomorphic vector fields Z_1 of M_1 and Z_2 of M_2 .

$n(\Gamma)$ is a real number. Denote by $\mathcal{F}(\Gamma, t_\Gamma^\kappa)$ the set of Fourier integral operators associated to $(\Gamma, t_\Gamma^\kappa)$.

Theorem 5.1. *The map $\mathcal{F}(\Gamma, t_\Gamma^\kappa) \rightarrow C^\infty(\Gamma)[[\hbar]]$ which sends an operator into its total symbol is well-defined and onto. Its kernel consists of the operators $O(k^{-\infty})$.*

The *principal symbol* of $T_k \in \mathcal{F}(\Gamma, t_\Gamma^\kappa)$ is the first coefficient $g_0 \in C^\infty(\Gamma)$ of the total symbol. If it doesn't vanish, T_k is said *elliptic*. In [7], we proved the basic results regarding the composition properties of this type of Fourier integral operators.

5.2. Toeplitz operators. The first example of Fourier integral operators are the Toeplitz operators. The diagonal Δ_r is a Lagrangian submanifold of $M_r \times M_r^-$. Denote by $t_{\Delta_r}^\kappa$ the flat section of $L_r^\kappa \boxtimes L_r^{-\kappa} \rightarrow \Delta_r$ such that

$$t_{\Delta_r}^\kappa(x, x) = z \otimes z^{-1} \text{ if } z \in L_x^\kappa \text{ and } z \neq 0.$$

Definition 5.2. \mathcal{F}_r is the space of Fourier integral operators associated to $(\Delta_r, t_{\Delta_r}^\kappa)$ with $n(\Delta_r) = n_r$ the complex dimension of M_r .

By identifying Δ_r with M_r , we consider the total symbols of these operators as formal series of $C^\infty(M_r)[[\hbar]]$. Our main result in [6] was the following theorem.

Theorem 5.3. Every Toeplitz operator (T_k) of M_r is a Fourier integral operator associated to $(\Delta_r, t_{\Delta_r}^\kappa)$ and conversely. Furthermore, there exists an equivalence of star-products

$$E : C^\infty(M_r)[[\hbar]] \rightarrow C^\infty(M_r)[[\hbar]]$$

such that if (T_k) is a Toeplitz operator with contravariant symbol $\sum \hbar^l f_l$, then the total symbol of (T_k) as a Fourier integral operator is $E(\sum \hbar^l f_l)$.

The same result holds for the Toeplitz operators of M . In the following we use the notations Δ , t_Δ and \mathcal{F} corresponding to Δ_r , $t_{\Delta_r}^\kappa$ and \mathcal{F}_r on M .

5.3. The λ -Toeplitz operators. Recall some notations of section 3. The action of $\theta \in \mathbb{T}^d$ on M (resp. L^κ) is denoted by l_θ (resp. \mathcal{L}_θ). P is the level set $\mu^{-1}(\lambda)$ and p is the projection $P \rightarrow M_r$.

Let Λ be the moment Lagrangian

$$\Lambda = \{(l_\theta.x, x); \quad x \in P \text{ and } \theta \in \mathbb{T}^d\}$$

introduced by Weinstein in [25]. Λ is a Lagrangian manifold of $M \times M^-$. Let t_Λ^κ be the section of $L^\kappa \boxtimes L^{-\kappa} \rightarrow \Lambda$ such that

$$t_\Lambda^\kappa(l_\theta.x, x) = \mathcal{L}_\theta.z \otimes z^{-1} \quad \text{if } z \in L_x^\kappa \text{ and } z \neq 0$$

This is a flat section with constant norm equal to 1.

Definition 5.4. \mathcal{F}_λ is the set of Fourier integral operators T_k associated to Λ and t_Λ^κ with $n(\Lambda) = n - \frac{d}{2}$ and such that

$$\mathcal{L}_\theta^* T_k = T_k \mathcal{L}_\theta^* = T_k, \quad \forall \theta$$

or equivalently $\Pi_{\lambda,k} T_k \Pi_{\lambda,k} = T_k$.

We will deduce the following result from the fact that the algebra \mathcal{T}_λ is isomorphic to the algebra \mathcal{T}_r of Toeplitz operators of M_r (theorems 4.24 and 4.25).

Theorem 5.5. \mathcal{F}_λ is the algebra \mathcal{T}_λ of λ -Toeplitz operators.

A similar characterization was given by Guillemin-Sternberg for pseudodifferential operators in [14]. Their proof starts from the fact that $\Pi_{\lambda,k}$ is a Fourier integral operator of \mathcal{F}_λ . If T_k is a Toeplitz operator of M , then the symbolic calculus of Fourier integral operators implies that $\Pi_{\lambda,k} T_k \Pi_{\lambda,k}$ belongs to \mathcal{F}_λ . This follows essentially from the composition of canonical relations

$$\Lambda \circ \Delta \circ \Lambda = \Lambda.$$

In the same way, we can show that \mathcal{F}_λ is a $*$ -algebra (cf. theorem 4.5), define the principal symbol and prove theorem 18. The corresponding compositions of canonical relations are

$$\Lambda \circ \Lambda = \Lambda, \quad \Lambda^t = \Lambda.$$

The difficulty of this approach is that it uses the properties of composition of the Fourier integral operators. In addition, the composition of Λ with itself is not transverse. But it has the advantage to be more general and can be transposed in other contexts.

5.4. The Guillemin-Sternberg isomorphism and its unitarization. Let Θ be the Lagrangian submanifold of $M_r \times M^-$

$$\Theta = \{(p(x), x); \quad x \in P\}$$

Recall that $L_r^\kappa \rightarrow M_r$ is the quotient of $L^\kappa \rightarrow P$ by the action of \mathbb{T}^d . If $z \in L_x^\kappa$ where $x \in P$, we denote by $[z] \in L_{r,p(x)}^\kappa$ its equivalence class. Then define the section t_Θ^κ of $L_r^\kappa \boxtimes L^{-\kappa} \rightarrow \Theta$ by

$$t_\Theta^\kappa(p(x), x) = [z] \otimes z^{-1} \quad \text{if } z \in L_x^\kappa \text{ and } z \neq 0.$$

It is flat with constant norm equal to 1.

Definition 5.6. $\mathcal{F}_{\lambda,r}$ is the space of Fourier integral operators T_k associated to $(\Theta, t_\Theta^\kappa)$ with $n(\Theta) = n - \frac{3}{4}d$ and such that

$$T_k \mathcal{L}_\theta^* = T_k, \quad \forall \theta$$

or equivalently $T_k \Pi_{\lambda,k} = T_k$.

Theorem 5.7. The Guillemin-Sternberg isomorphism V_k and the unitary operator $U_k = V_k(V_k^* V_k)^{-\frac{1}{2}}$ are elliptic operators of $\mathcal{F}_{\lambda,r}$.

This result is coherent with the fact that U_k induces an isomorphism between the algebra of the λ -Toeplitz operators and the algebra of the Toeplitz operators of M_r . Indeed, observe that

$$\Theta \circ \Lambda \circ \Theta^t = \Delta_r, \quad \Theta^t \circ \Delta_r \circ \Theta = \Lambda$$

which corresponds to the equalities

$$U_k T_k U_k^* = S_k, \quad U_k^* S_k U_k = T_k$$

where T_k is a λ -Toeplitz operator and S_k the reduced Toeplitz operator.

To prove theorems 5.5 and 5.7, we first explain how the spaces $\mathcal{F}_{\lambda,r}$, \mathcal{F}_λ , \mathcal{F}_r of Fourier integral operators are related by the Guillemin-Sternberg isomorphism. Then we deduce theorems 5.5 and 5.7 from theorem 4.24.

5.5. The relations between \mathcal{F}_λ , $\mathcal{F}_{\lambda,r}$ and \mathcal{F}_r . Since we consider Fourier integral operators which are equivariant, we need an equivariant version of theorem 5.1. Let us consider the same data as in section 5.1. Let G be a compact Lie group which acts on $M_1 \times M_2$ preserving the Kähler structure of $M_1 \times M_2^-$. Assume that this action lifts to $L_1^\kappa \boxtimes L_2^{-\kappa}$, preserving the Hermitian structure and connection.

Suppose that the Lagrangian manifold Γ and the section t_Γ^κ is G -invariant. Let us denote by $\mathcal{F}_G(\Gamma, t_\Gamma^\kappa)$ the space of Fourier integral operators associated to $(\Gamma, t_\Gamma^\kappa)$ whose kernel is G -invariant.

Theorem 5.8. The total symbol of an operator $T_k \in \mathcal{F}_G(\Gamma, t_\Gamma^\kappa)$ is G -invariant. Furthermore the total symbol map $\mathcal{F}_G(\Gamma, t_\Gamma^\kappa) \rightarrow C_G^\infty(\Gamma)[[\hbar]]$ is onto.

Proof. Let $T_k \in \mathcal{F}_G(\Gamma, t_\Gamma^\kappa)$. Its kernel is of the form (26) on a neighborhood of Γ . Hence,

$$T_k(x_1, x_2) = \left(\frac{k}{2\pi}\right)^{n(\Gamma)} f(x_1, x_2, k) t_\Gamma^\kappa(x_1, x_2) + O(k^{-\infty})$$

for every $(x_1, x_2) \in \Gamma$. It follows that the total symbol is G -invariant. Conversely, if the total symbol is G -invariant, we can define a G -invariant kernel of the form (26) with a G -invariant neighborhood U , a G -invariant section E_Γ^κ and a G -invariant sequence $f(\cdot, k)$. The operator obtained T'_k does not necessarily satisfy

$$\Pi_k^1 T'_k \Pi_k^2 = T'_k.$$

So we set

$$T_k = \Pi_k^1 T'_k \Pi_k^2.$$

Using that Π_k^1 and Π_k^2 are Fourier integral operators associated to the diagonal of M_1 and M_2 respectively, we prove that the kernels of T_k and T'_k are the same modulo $O_\infty(k^{-\infty})$. Consequently T_k belongs to $\mathcal{F}_G(\Gamma, t_\Gamma^\kappa)$ and has the required symbol. \square

Corollary 5.9. *There is a natural identification between total symbols of operators of \mathcal{F}_λ (resp. $\mathcal{F}_{\lambda,r}$) and formal series of $C^\infty(M_r)[[\hbar]]$.*

Proof. By theorem 5.8, the total symbols of the operators of \mathcal{F}_λ are the formal series of $C^\infty(\Lambda)[[\hbar]]$ invariant with respect to the action of $\mathbb{T}^d \times \mathbb{T}^d$ on $\Lambda \subset M^2$. The map

$$\Lambda \rightarrow M_r, \quad (y, x) \rightarrow p(x)$$

is a $(\mathbb{T}^d \times \mathbb{T}^d)$ -principal bundle. So there is a one-to-one correspondence between $C_{\mathbb{T}^d \times \mathbb{T}^d}^\infty(\Lambda)$ and $C^\infty(M_r)$. The proof is the same for the total symbols of the operators of $\mathcal{F}_{\lambda,r}$. \square

Theorem 5.10. *The following maps*

$$\begin{aligned} \mathcal{F}_\lambda &\rightarrow \mathcal{F}_{\lambda,r}, & T_k &\rightarrow V_k T_k \\ \mathcal{F}_\lambda &\rightarrow \mathcal{F}_r, & T_k &\rightarrow V_k T_k V_k^* \end{aligned}$$

are well-defined and bijective. Furthermore, if $T_k \in \mathcal{F}_\lambda$, the total symbols of $V_k T_k$, $V_k T_k V_k^$ and T_k are the same with the identifications of corollary 5.9.*

Proof. These properties follows immediately from the definition of the Fourier integral operators. Indeed consider two operators

$$T_k : \mathcal{H}_{\lambda,k} \rightarrow \mathcal{H}_{\lambda,k}, \quad S_k : \mathcal{H}_{\lambda,k} \rightarrow \mathcal{H}_{r,k}.$$

Extend them to the space of L^2 sections in such a way that they vanish on the orthogonal of $\mathcal{H}_{\lambda,k}$. Then $S_k = V_k T_k$ or equivalently $T_k = W_k S_k$ if and only if the kernels of T_k and S_k satisfy

$$(27) \quad T_k = \left(\frac{k}{2\pi}\right)^{\frac{d}{4}} (p_\mathbb{C} \boxtimes \text{Id})^* S_k \quad \text{over } P_\mathbb{C} \times M.$$

This follows from the definition of V_k (cf. definition 3.7). Furthermore by proposition 4.9, the kernel of T_k vanishes over $P_\mathbb{C}^c \times M$. So we may recover the kernel of T_k from the kernel of S_k and conversely. Using this we can directly check that $T_k \in \mathcal{F}_\lambda$ if and only if $S_k \in \mathcal{F}_{\lambda,r}$.

To do this observe that the data which define the operators of \mathcal{F}_λ and $\mathcal{F}_{\lambda,r}$ are related in the following way:

$$(p \times \text{Id})^{-1}(\Theta) = \Lambda, \quad (p \boxtimes \text{Id})^* t_\Theta^\kappa = t_\Lambda^\kappa$$

Using that $p_\mathbb{C} : P_\mathbb{C} \rightarrow M_r$ is a holomorphic map, it comes that

$$(p_\mathbb{C} \boxtimes \text{Id})^* E_\Theta^\kappa = E_\Lambda^\kappa.$$

That $T_k \rightarrow V_k T_k$ is well defined, bijective and preserves the total symbols follows easily.

For the second map, we can proceed in a similar way. Let S_k be such that $\Pi_{r,k} S_k \Pi_{r,k} = S_k$. Then computing successively the Schwartz kernels of $W_k S_k$, $(W_k S_k)^*$, $W_k(W_k S_k)^*$ and using

$$(W_k(W_k S_k)^*)^* = W_k S_k W_k^*,$$

we obtain that the kernels of $T_k = W_k S_k W_k^*$ and S_k satisfy

$$T_k(x, y) = \left(\frac{k}{2\pi}\right)^{\frac{d}{2}} e^{-k\varphi(y)} (p_{\mathbb{C}} \boxtimes p_{\mathbb{C}})^* S_k(x, y) \quad \text{over } P_{\mathbb{C}} \times P_{\mathbb{C}}.$$

where the function φ has been defined in proposition 4.8. \square

5.6. Proofs of theorems 5.5 and 5.7. Recall that by theorem 4.24, there is a bijection from \mathcal{T}_λ onto \mathcal{T}_r

$$(28) \quad \mathcal{T}_\lambda \rightarrow \mathcal{T}_r, \quad T_k \rightarrow W_k^* T_k W_k.$$

Furthermore, we know that $\mathcal{T}_r = \mathcal{F}_r$ by theorem 5.3. In theorem 5.10, we proved that the map

$$(29) \quad \mathcal{F}_\lambda \rightarrow \mathcal{F}_r, \quad T_k \rightarrow V_k T_k V_k^*$$

is a bijection. Let us deduce that $\mathcal{T}_\lambda = \mathcal{F}_\lambda$.

Let T_k belong to \mathcal{F}_λ . By (29), $V_k T_k V_k^*$ belongs to \mathcal{T}_r . By (28), $W_k^* T_k W_k$ belongs to \mathcal{T}_r . Since the product of Toeplitz operators is a Toeplitz operator,

$$(W_k^* W_k)(V_k T_k V_k^*)(W_k^* W_k) = W_k^* T_k W_k$$

belongs to \mathcal{T}_r . By (28) T_k belongs to \mathcal{T}_λ .

Conversely assume that T_k belongs to \mathcal{T}_λ . By (28), $W_k^* T_k W_k$ belongs to \mathcal{T}_r . Write

$$V_k T_k V_k^* = (V_k V_k^*)(W_k^* T_k W_k)(V_k V_k^*)$$

Observe that $V_k V_k^*$ is the inverse of $W_k^* W_k$ in the sense of Toeplitz operators, i.e.

$$\begin{aligned} \Pi_{r,k}(V_k V_k^*)\Pi_{r,k} &= V_k V_k^*, \\ (V_k V_k^*)(W_k^* W_k) &= (W_k^* W_k)(V_k V_k^*) = \Pi_{r,k}. \end{aligned}$$

Since $(W_k^* W_k)$ is a Toeplitz operator with a non-vanishing symbol by theorem 4.24, $V_k V_k^*$ is a Toeplitz operator. Consequently $V_k T_k V_k^*$ belongs to \mathcal{T}_r and by (29), T_k belongs to \mathcal{F}_λ .

Let us prove theorem 5.7. By theorem 5.5, $\Pi_{\lambda,k}$ belongs to \mathcal{F}_λ . So theorem 5.10 implies that $V_k = V_k \Pi_{\lambda,k}$ belongs to $\mathcal{F}_{\lambda,r}$. Let us consider now U_k . We have

$$U_k = (W_k^* W_k)^{-\frac{1}{2}} W_k^* = V_k (W_k (W_k^* W_k)^{-\frac{1}{2}} W_k^*)$$

As we saw in the proof of theorem 4.25, $(W_k^* W_k)^{-\frac{1}{2}}$ belongs to \mathcal{T}_r . So by theorem 4.24, $W_k (W_k^* W_k)^{-\frac{1}{2}} W_k^*$ belongs to $\mathcal{T}_\lambda = \mathcal{F}_\lambda$. And theorem 5.10 implies that U_k belongs to $\mathcal{F}_{\lambda,r}$.

6. TOEPLITZ OPERATORS ON ORBIFOLD

In this part, we prove the basic results about the Toeplitz operators on the orbifold M_r . We describe their kernels as Fourier integral operators associated to the diagonal, prove that the set of Toeplitz operators is an algebra and describe the associated symbolic calculus. Finally we compute the asymptotic of the density of states of a Toeplitz operator.

6.1. Schwartz kernel on orbifold. Let us introduce some notations and state some basic facts about kernels of operators on orbifolds. Let X be a reduced orbifold with a vector bundle $E \rightarrow X$. So every chart $(|U|, U, G, \pi_U)$ of X is endowed with a G -bundle $E_U \rightarrow U$. G acts effectively on U . If $g \in G$, we denote by $a_g : U \rightarrow U$ its action on U and by $\mathcal{A}_g : E_U \rightarrow E_U$ its lift. If s is a section of $E \rightarrow X$, we denote by s_U the corresponding invariant section of E_U .

As in the manifold case, we can define the dual bundle of E , the tensor product of two bundles over X , the orbifold X^2 and the bundle $E \boxtimes E^* \rightarrow X^2$. Let δ be a volume form of X . Then every section T of $E \boxtimes E^*$ defines an operator T in the following way. Consider two charts $(|U|, U, G, \pi_U)$ and $(|V|, V, H, \pi_V)$ of X . Then T is given over $|U| \times |V|$ by a $(G \times H)$ -invariant section T_{UV} of $E_U \boxtimes E_V^*$. If s is a section of E with compact support in $|V|$, then

$$(Ts)_U = \frac{1}{\#H} \int_V T_{UV} \cdot s_V \delta_V.$$

Assume that E is Hermitian and define the scalar product of sections of E by using δ . If T is an operator which acts on $C^\infty(X, E)$, vanishing over the orthogonal of a finite dimensional subspace of $C^\infty(X, E)$, it is easily proved that T has a Schwartz kernel T . It is unique. Furthermore, if E has rank one, the trace of T is given by

$$\text{Tr}(T) = \int_X \Delta^* T \delta$$

where $\Delta : X \rightarrow X^2$ is the diagonal map. Since Δ is a good map (in the sense of [8]), the pull-back $\Delta^*(E \boxtimes E^*)$ is well-defined. It is naturally isomorphic to $E \otimes E^* \rightarrow X$. So $\Delta^* T$ is a section of $E \otimes E^* \simeq \mathbb{C}$. Finally the previous integral is defined with the orbifold convention: if $(|U|, U, G, \pi_U)$ is a chart of X and $\Delta^* T$ has support in $|U|$, it is given by

$$\frac{1}{\#G} \int_U (\Delta^* T)_U \delta_U.$$

Note also that it is false that every section of $E \otimes E^*$ is the pull-back by Δ of a section of $E \boxtimes E^*$. For instance when E is a line bundle, $E \otimes E^* \simeq \mathbb{C}$ has nowhere vanishing sections, whereas it may happen that every section of $E \boxtimes E^*$ vanishes at some point (x, x) of the diagonal. This explains some complications in the description of the kernels of Toeplitz operators.

6.2. The algebra \mathcal{F}_r . Recall that (M_r, ω_r) is a compact Kähler reduced orbifold with a prequantum bundle $L_r^\kappa \rightarrow M_r$ whose curvature is $-i\kappa\omega_r$. Let us consider a family $(T_k)_k$ of operators, with Schwartz kernels

$$T_k \in C^\infty(M_r^2, L_r^k \boxtimes L_r^{-k}), \quad k = \kappa, 2\kappa, 3\kappa, \dots$$

As in the manifold case (cf. section 5.2), the definition of the operators of \mathcal{F}_r consists in two parts. The first assumption is

Assumption 6.1. T_k is $O_\infty(k^{-\infty})$ on every compact set $K \subset M_r^2$ such that $K \cap \Delta_r = \emptyset$.

The description of T_k on a neighborhood of the diagonal doesn't generalize directly from the manifold case, because the definition of the section $E_{\Delta_r}^\kappa$ doesn't make sense. Fortunately we can keep the same ansatz on the orbifold charts. If $(|U|, U, G, \pi_U)$ is an orbifold chart of M_r , we assume that:

Assumption 6.2. There exists a section $T'_{k,U}$ of $L_{r,U}^k \boxtimes L_{r,U}^{-k}$ invariant with respect to the diagonal action of G and of the form

$$(30) \quad T'_{k,U}(x, y) = \left(\frac{k}{2\pi}\right)^{n_r} E_{\Delta_U}^k(x, y) f(x, y, k) + O_\infty(k^{-\infty})$$

on a neighborhood of the diagonal Δ_U , where $E_{\Delta_U}^\kappa$ and $f(., k)$ satisfy the assumptions (26. i) and (26. ii) with $(\Gamma, t_\Gamma^\kappa) = (\Delta_U, t_{\Delta_U}^\kappa)$, such that

$$(31) \quad T_{k,UU} = \sum_{g \in G} (\mathcal{A}_g \boxtimes \text{Id})^* T'_{k,U}.$$

Observe that we can use $T'_{k,U}$ instead of $T_{k,UU}$ to compute $(T_k \Psi)_U$ when Ψ has compact support in U :

$$(32) \quad \begin{aligned} (T_k \Psi)_U(x) &= \frac{1}{\#G} \int_U T_{k,UU}(x, y) \Psi_U(y) \delta_U(y). \\ &= \int_U T'_{k,U}(x, y) \Psi_U(y) \delta_U(y). \end{aligned}$$

This follows from (31) and the fact that Ψ_U is G -invariant and $T'_{k,U}$ is invariant with respect to the diagonal action.

Definition 6.3. \mathcal{F}_r is the set of operators (T_k) such that

$$\Pi_{r,k} T_k \Pi_{r,k} = T_k$$

and whose Schwartz kernel satisfies assumptions 6.1 and 6.2 for every orbifold chart $(|U|, U, G, \pi_U)$.

The basic result is the generalization of the theorem of Boutet de Monvel and Sjöstrand on the Szegö projector.

Theorem 6.4. The projector Π_k is an operator of \mathcal{F}_r .

This theorem will be proved in section 7. Let us deduce from it the properties of \mathcal{F}_r . First we define the total symbol map

$$\sigma : \mathcal{F}_r \rightarrow C^\infty(M_r)[[\hbar]],$$

which sends T_k into the formal series $\sum \hbar^l g_l$ such that

$$(33) \quad T_k(x, x) = \left(\frac{k}{2\pi}\right)^{n_r} \sum k^{-l} g_l(x) + O(k^{-\infty})$$

for every x which belongs to the principal stratum of M_r .

Proposition 6.5. σ is well-defined and onto. Furthermore $\sigma(T_k) = 0$ if and only if T_k is smoothing in the sense of section 5.1.1, i.e. its kernel is $O_\infty(k^{-\infty})$.

Proof. First let us prove that σ is well-defined. By (31),

$$T_{k,UU}(x, x) = \sum_{g \in G} (\mathcal{A}_g \otimes \text{Id})^* T'_{k,U}(a_g \cdot x, x)$$

Assume that $\pi_U(x)$ belongs to the principal stratum of M_r . So if $g \neq \text{id}_G$, $a_g \cdot x \neq x$ which implies that $T'_{k,U}(a_g \cdot x, x) = O(k^{-\infty})$. Consequently (30) leads to

$$T_{k,UU}(x, x) = \left(\frac{k}{2\pi}\right)^{n_r} f(x, x, k) + O(k^{-\infty}).$$

This proves the existence of the asymptotic expansion (33) and that the g_l extend to C^∞ functions on M_r . Furthermore, since the principal stratum is dense in M_r , these functions are uniquely determined by the kernel T_k . If T_k is smoothing, they vanish. Conversely, if the functions g_l vanish, $T'_{k,U}$ is $O_\infty(k^{-\infty})$ and the same holds for $T_{k,UU}$. So the kernel of σ consists of the T_k such that T_k is $O_\infty(k^{-\infty})$.

Let us prove that σ is onto. Let $\sum \hbar^l g_l$ be a formal series of $C^\infty(M_r)[[\hbar]]$. First we construct a Schwartz kernel T_k satisfying assumptions 6.1, 6.2 and (33). To do this, we introduce on every orbifold chart a kernel $T'_{k,U}$ of the form (30) where the functions $f(., k)$ are such that

$$f(., k) = \sum k^{-l} g_{l,U} + O(k^{-\infty})$$

with the $g_{l,U} \in C^\infty(U)$ corresponding to the g_l . The existence of $T'_{k,U}$ is a consequence of the Borel lemma as in the manifold case. Furthermore since the functions $g_{l,U}$ are G -invariant and the diagonal action of G preserves Δ_U and $t_{\Delta_U}^\kappa$, we can obtain a G -invariant section $T'_{k,U}$.

Then we piece together the $T_{k,UU}$ by using a partition of unity subordinate to a cover of M_r by orbifold charts. To do this, we have to check that two kernels T_{k,U_1U_1} and T_{k,U_2U_2} on two orbifold charts

$$(|U_1|, U_1, G_1, \pi_{U_1}), \quad (|U_2|, U_2, G_2, \pi_{U_2})$$

define the same section over $|U_1| \cap |U_2|$ modulo $O_\infty(k^{-\infty})$. Recall that the compatibility between orbifold charts is expressed by using orbifold charts $(|U|, U, G, \pi_U)$ which embed into $(|U_i|, U_i, G_i, \pi_{U_i})$. Denote by $\rho_i : G \rightarrow G_i$ the injective group homomorphism and by $j_i : U \rightarrow U_i$ the ρ_i -equivariant embeddings. We have

$$(34) \quad \forall g \in G_i, \forall x \in j_i(U), \quad a_g \cdot x \in j_i(U) \Rightarrow g \in \rho_i(G).$$

We have to prove that

$$(35) \quad (j_i \boxtimes j_i)^* T_{k,U_iU_i} = T_{k,UU} + O_\infty(k^{-\infty})$$

By (34), if $g \in G_i - \rho_i(G)$, then $a_g(j_i(U)) \cap j_i(U) = \emptyset$. Since T'_{k,U_i} is $O_\infty(k^{-\infty})$ outside the diagonal, this implies

$$T_{k,U_iU_i}|_{j_i(U) \times j_i(U)} = \sum_{g \in \rho_i(G)} (\mathcal{A}_g \boxtimes \text{Id})^* T'_{k,U_i}|_{j_i(U) \times j_i(U)} + O_\infty(k^{-\infty})$$

Furthermore, since $j_i^* g_{l,U_i} = g_{l,U}$, we have

$$(j_i \boxtimes j_i)^* T'_{k,U_i} = T'_{k,U} + O_\infty(k^{-\infty})$$

Both of the previous equation lead to (35) by using that the map j_i are ρ_i -equivariant.

The section T_k obtained is the Schwartz kernel of an operator T_k . We don't have necessarily $\Pi_k T_k \Pi_k = T_k$, but only

$$(36) \quad \Pi_k T_k \Pi_k \equiv T_k$$

modulo an operator whose kernel is $O_\infty(k^{-\infty})$. So we replace T_k with $\Pi_k T_k \Pi_k$. The proof of (36) is a consequence of theorem 6.4 and is similar to the manifold case. Actually, if two operators R_k and S_k satisfy assumptions 6.1 and 6.2, the kernel T_k of their product is given on a orbifold chart by

$$T_{k,UU} = \sum_{g \in G} (\mathcal{A}_g \boxtimes \text{Id})^* T'_{k,U}$$

where

$$(37) \quad T'_{k,U}(x, y) = \int_U S'_{k,U}(x, z) \cdot R'_{k,U}(z, y) \delta_U(z) + O_\infty(k^{-\infty})$$

This follows from equation (32). \square

Theorem 6.6. \mathcal{F}_r is a $*$ -algebra and the induced product of total symbols is a star-product of $C^\infty(M_r)[[\hbar]]$. Every operator of \mathcal{F}_r is of the form

$$(38) \quad \Pi_{r,k} M_{f(.,k)} \Pi_{r,k} + O(k^{-\infty})$$

where $f(.,k)$ is a symbol of $S(M_r)$ and conversely. The map

$$E : C^\infty(M_r)[[\hbar]] \rightarrow C^\infty(M_r)[[\hbar]]$$

which sends the formal series $\sum \hbar^l f_l$ corresponding to the multiplier $f(.,k)$ into the total symbol of the operator (38), is well defined. It is an equivalence of star-products.

Proof. As we have seen in the previous proof, the composition of kernels of operators of \mathcal{F}_r corresponds in a orbifold chart to a composition on a manifold (cf. equation (37)). So the proof in the manifold case [6] extends directly. Let us recall the main steps. We first prove that \mathcal{F}_r is a $*$ -algebra and compute the product of the total symbols by applying the stationary phase lemma to the composition of the kernels. In the same way, we can compute the kernel of the operator (38), since $\Pi_{r,k}$ belongs to \mathcal{F}_r by theorem 6.4. As a result of the computation, this operator belongs to \mathcal{F}_r and we obtain that the map E is an equivalence of star-product. From this we deduce that conversely every operator of \mathcal{F}_r is of the form (38). \square

6.3. Spectral density of a Toeplitz operator. Let T_k be a self-adjoint Toeplitz operator over M_r . Denote by d_k the dimension of $\mathcal{H}_{r,k}$ and by $E_1 \leq E_2 \leq \dots \leq E_{d_k}$ the eigenvalues of T_k . The spectral density of T_k is the measure of \mathbb{R}

$$\mu_{T_k}(E) = \sum_{i=1}^{d_k} \delta(E - E_i)$$

Let f be a C^∞ function on \mathbb{R} . We will estimate $\langle \mu_{T_k}, f \rangle$. The first step is to compute the operator $f(T_k)$.

Theorem 6.7. *If f is a C^∞ function on \mathbb{R} , then $f(T_k)$ is a Toeplitz operator. Furthermore if g_0 is the principal symbol of T_k then $f(g_0)$ is the principal symbol of $f(T_k)$.*

The proof is similar to the manifold case (cf. proposition 12 of [6]). Now we have

$$\langle \mu_{T_k}, f \rangle = \sum_{i=1}^{d_k} f(E_i) = \text{Tr } f(T_k)$$

By the previous theorem, it suffices to estimate the trace of a Toeplitz operator. Recall that

$$\text{Tr } T_k = \int_{M_r} T_k(x, x) \delta_{M_r}(x)$$

Let us begin with the computation in a orbifold chart $(|U|, U, G, \pi_U)$. Let η be a C^∞ function of M_r whose support is included in $|U|$. It follows from assumption 6.2 that

$$\int_{M_r} \eta(x) T_k(x, x) \delta_{M_r}(x) = \frac{1}{\#G} \sum_{g \in G} I(g, U)$$

where

$$I(g, U) = \int_U \eta_U(x) (\mathcal{A}_g \boxtimes \text{Id})^* T'_{k,U}(x, x) \delta_U(x).$$

Let U^g be the fixed point set of g ,

$$(39) \quad U^g = \{x \in U; a_g \cdot x = x\}.$$

Assume that U^g is connected. Then U^g is Kähler submanifold of U . Denote by $d(g)$ its complex codimension.

Let $y \in U^g$. $a_g : U \rightarrow U$ induces a linear transformation on the normal space $N_y = T_y U / T_y U^g$. It is a unitary map with eigenvalues

$$b_1(g), \dots, b_{d(g)}(g)$$

on the unit circle and not equal to 1. Furthermore, \mathcal{A}_g acts on the fiber of $L_{r,U}^\kappa$ at y by multiplication by $c^\kappa(g)$, where $c(g)$ is on the unit circle. Since U^g is connected, the complex numbers $b_i(g)$ and $c(g)$ do not depend on y .

Lemma 6.8. *The integral $I(g, U)$ admits an asymptotic expansion*

$$I(g, U) = \left(\frac{k}{2\pi}\right)^{n_r - d(g)} c(g)^{-k} \sum_{l=0}^{\infty} k^{-l} I_l(g, U) + O(k^{-\infty}).$$

The first coefficient is given by

$$I_0(g, U) = \left(\prod_{i=1}^{d(g)} (1 - b_i(g))^{-1} \right) \int_{U^g} \eta_U(x) f_{0,U}(x) \delta_{U^g}(x)$$

where f_0 is the principal symbol of T_k and δ_{U^g} is the Liouville measure of U^g .

Proof. Since $|T'_k(x, y)|$ is $O(k^{-\infty})$ if $x \neq y$, we can restrict the integral over a neighborhood of U^g . Let $y \in U^g$. Let (w^i) be a system of complex coordinates of U centered at y . Since g is of finite order, we can linearize the action of a_g . So we can assume that

$$a_g : (w_i)_{i=1, \dots, n_r} \rightarrow (w_1 b_1, \dots, w_d b_d, w_{d+1}, \dots, w_{n_r})$$

To simplify the notations we denoted by d, b_i the numbers $d(g), b_i(g)$. In the same way, we can choose a local holomorphic section s_r^κ of $L_{r,U}^\kappa$, which doesn't vanish on a neighborhood of y and such that

$$\mathcal{A}_g^* s_r^\kappa = c^{-\kappa}(g) s_r^\kappa$$

Let H_r be the function such that $(s_r^\kappa, s_r^\kappa) = e^{-\kappa H_r}$. Observe that $H_r(a_g.x) = H_r(x)$. Denote by t^κ the unitary section $e^{\kappa H/2} s^\kappa$. Introduce a function $\tilde{H}_r(x, y)$ defined on U^2 such that $\tilde{H}_r(x, x) = H_r(x)$ and

$$(0, \tilde{Z}).\tilde{H}_r \equiv 0 \text{ and } (Z, 0).\tilde{H}_r \equiv 0$$

modulo $O(|x - y|^\infty)$ for every holomorphic vector field Z of U . Using that

$$\nabla t^\kappa = \frac{\kappa}{2} (\bar{\partial} H - \partial H) \otimes t^\kappa$$

we compute the section $E_{\Delta_U}^k$ of (30) (cf. [6, proposition 1]), and obtain

$$T'_k(x, y) = \left(\frac{k}{2\pi}\right)^{n_r} e^{-k(\frac{1}{2}(H_r(x) + H_r(y)) - \tilde{H}_r(x, y))} f(x, y, k) t_r^k(x) \otimes t_r^{-k}(y)$$

Consequently the integral $I(g, U)$ is equal to

$$\left(\frac{k}{2\pi}\right)^{n_r} c^{-k}(g) \int_U e^{-k\phi_g(x)} f(a_g.x, x, k) \delta_U(x)$$

where $\phi_g(x) = H_r(x) - \tilde{H}_r(a_g.x, x)$. We estimate it by applying the stationary phase lemma. ϕ_g vanishes along U^g . Using that $a_g^* H_r = H_r$, we obtain for $i, j = 1, \dots, d$,

$$\partial_{w_i} \phi_g = \partial_{\bar{w}_i} \phi_g = \partial_{w_i} \partial_{w_j} \phi_g = \partial_{\bar{w}_i} \partial_{\bar{w}_j} \phi_g = 0$$

along U^g for $i, j = 1, \dots, d$. Denote by $H_{i,j}$ the second derivative $\partial_{w_i} \partial_{\bar{w}_j} H_r$. Then

$$\partial_{w_i} \partial_{\bar{w}_j} \phi_g = (1 - b_i) H_{i,j}$$

along U^g for $i, j = 1, \dots, d$. Furthermore

$$\delta_U(x) = \det[H_{i,j}]_{i,j=1, \dots, n_r} |dw_1 d\bar{w}_1 \dots dw_{n_r} d\bar{w}_{n_r}|$$

which leads to the result. \square

The next step is to patch together these local contributions. This involves a family of orbifolds associated to M_r which appears also in the Riemann-Roch theorem or in the definition of orbifold cohomology groups (cf. the associated orbifold of [19], the twisted sectors of [8], the inertia orbifold of [20]). The description of these orbifolds in the general case is rather complicated. Here, M_r is the quotient of P by a torus action, which simplifies the exposition (cf. the appendix of [15]).

Consider the following set

$$\tilde{P} = \{(x, g) \in P \times \mathbb{T}^d; l_g.x = x\}$$

To each connected component C of \tilde{P} is associated a element g of \mathbb{T}^d and a *support* $\bar{C} \subset P$ such that $C = \bar{C} \times \{g\}$. \bar{C} is a closed submanifold of P invariant with respect to the action of \mathbb{T}^d . The quotient

$$F := \bar{C}/\mathbb{T}^d$$

is a compact orbifold, which embeds into M_r . Since \mathbb{T}^d doesn't necessarily act effectively on \bar{C} , F is not in general a reduced orbifold. Denote by $m(F)$ its multiplicity.

Let $(x, g) \in C$ and denote by G the isotropy group of x . Let $U \subset P$ be a slice at x of the \mathbb{T}^d -action. Let $|U| = p(U)$ and π_U be the projection $U \rightarrow |U|$. Then $(|U|, U, G, \pi_U)$ is an orbifold chart of M_r . Introduce as in (39) the subset U^g of U and assume it is connected. Then $U^g = U \cap \bar{C}$. Let $|U^g| = p(U^g)$ and π_{U^g} be the projection $U^g \rightarrow |U^g|$. Then $(|U^g|, U^g, G, \pi_{U^g})$ is an orbifold chart of F . So $I_0(g, U)$ in lemma 6.8 is given by an integral over F . Furthermore, since F is connected, there exists complex numbers

$$b_1(g, F), \dots, b_{d(F)}(g, F) \text{ and } c(g, F)$$

on the unit circle corresponding to the numbers defined locally, with $d(F)$ the codimension of F in M_r . Observe also that F inherits a Kähler structure.

Denote by \mathcal{F} the set

$$\mathcal{F} := \{\bar{C}/\mathbb{T}^d; \bar{C} \times \{g\} \text{ is a component of } \tilde{P}\}$$

For every $F \in \mathcal{F}$, let \mathbb{T}_F^d be the set of $g \in \mathbb{T}^d$ such that $F = \bar{C}/\mathbb{T}^d$ and $\bar{C} \times \{g\}$ is a component of \tilde{P} . The point is that two components of \tilde{P} may have the same support. Since the set of components of \tilde{P} is finite, the various sets \mathcal{F} and \mathbb{T}_F^d are finite.

Theorem 6.9. *Let T_k be a self-adjoint Toeplitz operator on M_r with principal symbol g_0 . Let f be a C^∞ function on \mathbb{R} . Then $\langle \mu_{T_k}, f \rangle$ admits an asymptotic expansion of the form*

$$\sum_{F \in \mathcal{F}} \left(\frac{k}{2\pi}\right)^{n_r - d(F)} \sum_{g \in \mathbb{T}_F^d} c(g, F)^{-k} \sum_{l=0}^{\infty} k^{-l} I_l(F, g) + O(k^{-\infty})$$

where the coefficients $I_l(F, g)$ are complex numbers. Furthermore

$$I_0(F, g) = \frac{1}{m(F)} \left(\prod_{i=1}^{d(F)} (1 - b_i(g, F))^{-1} \right) \int_F f(g_0) \delta_F$$

where δ_F is the Liouville measure of F .

Remark 6.10. Each orbifold $F \in \mathcal{F}$ is the closure of a strata of M_r (cf. appendix of [15]). For instance, M_r itself belongs to \mathcal{F} and is the closure of the principal stratum of M_r . Note that $\mathbb{T}_{M_r}^d = \{0\}$ and that the other suborbifolds of \mathcal{F} have a positive codimension. So at first order,

$$\langle \mu_{T_k}, f \rangle = \left(\frac{k}{2\pi}\right)^{n_r} \int_{M_r} f(g_0) \delta_{M_r} + O(k^{n_r-1}).$$

Remark 6.11. Riemann-Roch-Kawasaki theorem gives the dimension of $\mathcal{H}_{r,k}$ in terms of characteristic forms, when k is sufficiently large:

$$\dim \mathcal{H}_{r,k} = \sum_{F \in \mathcal{F}} \sum_{g \in \mathbb{T}_F^d} \frac{1}{m(F)} \int_F \frac{\text{Td}(F) \text{Ch}(L_r^k, F, g)}{\text{D}(N_F, g)}$$

For the definition of these forms, we refer to theorem 3.3 of [19]. Let us compare this with the estimate of $\text{Tr}(\Pi_k)$ given by theorem 6.9. First, $\text{Ch}(L_r^k, F, g) \in \Omega(F)$ is a twisted characteristic form associated to the pull-back of L_r^k by the embedding $F \rightarrow M_r$. At first order

$$\text{Ch}(L_r^k, F, g) = c(g, F)^k \left(\frac{k}{2\pi} \right)^{n_F} \frac{\omega_F^{\wedge n_F}}{n_F!} + O(k^{n_F-1})$$

where $n_F = n_r - d(F)$ and ω_F is the symplectic form of F . $D(N_F, g)$ is a twisted characteristic form associated to the normal bundle N_F of the embedding $F \rightarrow M_r$,

$$D(N_F, g) \equiv \prod_{i=1}^{d(F)} (1 - b_i(g, F)) \mod \Omega^{\bullet \geq 2}(F)$$

$\text{Td}(F) \equiv 1$ modulo $\Omega^{\bullet \geq 2}(F)$ is the Todd form of F . Hence we recover the leading term $I_0(F, g)$ in theorem 6.9. \square

Remark 6.12. (harmonic oscillator). We can describe explicitly the various term of theorem 6.9 and deduce theorem 2.3. Denote by \mathcal{P} the set of greatest common divisors of the families $(\mathfrak{p}_i)_{i \in I}$ where I runs over the subsets of $\{1, \dots, n\}$. For every $\mathfrak{p} \in \mathcal{P}$, define the subset $I(\mathfrak{p})$ of $\{1, \dots, n\}$

$$I(\mathfrak{p}) := \{i; \mathfrak{p} \text{ divides } \mathfrak{p}_i\}$$

and the symplectic subspace $\mathbb{C}_{\mathfrak{p}}$ of \mathbb{C}^n

$$\mathbb{C}_{\mathfrak{p}} := \{z \in \mathbb{C}^n; z_i = 0 \text{ if } i \notin I(\mathfrak{p})\}.$$

Denote by $M_{\mathfrak{p}}$ the quotient of $P \cap \mathbb{C}_{\mathfrak{p}}$ by the S^1 -action. Then one can check the following facts: $\mathcal{F} = \{M_{\mathfrak{p}}; \mathfrak{p} \in \mathcal{P}\}$, the multiplicity of $M_{\mathfrak{p}}$ is \mathfrak{p} , its complex dimension is $\#I(\mathfrak{p}) - 1$. Furthermore, we have

$$S_{M_{\mathfrak{p}}}^1 = \{\zeta \in \mathbb{C}^*; \zeta^{\mathfrak{p}} = 1 \text{ and } \forall i \notin I(\mathfrak{p}), \zeta^{\mathfrak{p}_i} \neq 1\},$$

Note that the $S_{M_{\mathfrak{p}}}^1$ are mutually disjoint. Theorem 2.3 follows by using that

$$G = \cup_{\mathfrak{p} \in \mathcal{P}} S_{M_{\mathfrak{p}}}^1$$

and if $\zeta \in S_{M_{\mathfrak{p}}}^1$, then $c(\zeta, M_{\mathfrak{p}}) = \zeta$ and the $b_i(\zeta, M_{\mathfrak{p}})$ are the $\zeta^{\mathfrak{p}_i}$ with $i \notin I(\mathfrak{p})$.

For example, if $(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3) = (2, 4, 3)$, then $\mathcal{P} = \{1, 2, 4, 3\}$. There are four supports: $M_1 = M_r$, M_2 which is 1-dimensional and M_4, M_3 which consist of one point. The subsets of S^1 associated to these supports are

$$S_1^1 = \{1\}, S_2^1 = \{-1\}, S_4^1 = \{i, -i\} \text{ and } S_3^1 = \{e^{i\pi/3}, e^{i2\pi/3}\}. \quad \square$$

7. THE KERNEL OF THE SZEGÖ PROJECTOR OF M_r

In this section, we prove that the Szegő projector $\Pi_{r,k}$ of the orbifold M_r is a Fourier integral operator, which is the content of theorem 6.4. In the manifold case this result is a consequence of a theorem of Boutet de Monvel and Sjöstrand on the kernel of the Szegő projector associated to the boundary of a strictly pseudoconvex domain [5]. We won't adapt the proof of Boutet de Monvel and Sjöstrand. Instead we will deduce this result from the same result for the Szegő projector of M , i.e. we will show that

$$\Pi_k \in \mathcal{F} \Rightarrow \Pi_{r,k} \in \mathcal{F}_r.$$

First we define the algebra \mathcal{F}_{λ} , that we introduced in section 5 for a free torus action. Then following an idea of Guillemin and Sternberg (cf. appendix of [13]), we prove that $\Pi_{\lambda,k} \in \mathcal{F}_{\lambda}$. Then we relate the algebras \mathcal{F}_{λ} and \mathcal{F}_r as in theorem 5.10 and deduce theorem 6.4. To compare with section 5, the proof of theorem 5.5 was in reverse order, i.e. we showed that $\Pi_{r,k} \in \mathcal{F}_r \Rightarrow \Pi_{\lambda,k} \in \mathcal{F}_{\lambda}$.

We think that the ansatz we propose for the kernel of $\Pi_{r,k}$ is also valid for a general Kähler orbifold, not obtained by reduction. But here, it is easier and more natural to deduce this by reduction.

7.1. The algebra \mathcal{F}_λ . Since the \mathbb{T}^d -action is not necessarily free, we have to modify the definition of the algebra \mathcal{F}_λ . We consider M as an orbifold and give local ansatz in orbifold charts as we did with the Toeplitz operators in section 6.2.

Introduce as in remark 3.8 an orbifold chart $(|U|, U, G, \pi_U)$ of M_r with the associated orbifold chart $(|V|, V, G, \pi_V)$ of $P_{\mathbb{C}}$. Denote by a_g (resp. l_θ) the action of $g \in G$ (resp. $\theta \in \mathbb{T}^d$) on V . These actions both lift to $L_V^\kappa := \pi_V^* L^\kappa$. We denote them by \mathcal{A}_g and \mathcal{L}_θ .

Define the data $(\Lambda_V, t_{\Lambda_V}^\kappa)$ corresponding to the data $(\Lambda, t_\Lambda^\kappa)$ of section 5.3.

$$\Lambda_V := \{(\theta, 0, u, \theta', 0, u) \in V^2; \theta, \theta' \in \mathbb{T}^d \text{ and } u \in U\}$$

$$t_{\Lambda_V}^\kappa(\theta, 0, u, \theta', 0, u) := \mathcal{L}_{\theta-\theta'} z \otimes z^{-1} \quad \text{if } z \in L_{V,(\theta',0,u)}^\kappa \text{ and } z \neq 0$$

Note that $t_{\Lambda_V}^\kappa$ is well-defined because the \mathbb{T}^d -action on V is free. In the definition of an operator T_k of \mathcal{F}_λ , we will assume that the lift $T_{k,VV}$ of its Schwartz kernel T_k to V^2 satisfies

Assumption 7.1. *There exists a section $T'_{k,V}$ of $L_V^k \boxtimes L_V^{-k}$ invariant with respect to the action of $\mathbb{T}^d \times \mathbb{T}^d$ and the diagonal action of G and of the form*

$$(40) \quad T'_{k,V}(x, y) = \left(\frac{k}{2\pi}\right)^{n-\frac{d}{2}} E_{\Lambda_V}^k(x, y) f(x, y, k) + O_\infty(k^{-\infty})$$

on a neighborhood of Λ_V , where $E_{\Lambda_V}^k$ and $f(\cdot, \cdot, k)$ satisfy the assumptions (26. i) and (26. ii) with $(\Gamma, t_\Gamma^\kappa) = (\Lambda_V, t_{\Lambda_V}^\kappa)$, such that

$$T_{k,VV} = \sum_{g \in G} (\mathcal{A}_g \boxtimes \text{Id})^* T'_{k,V}.$$

The whole definition of an operator of \mathcal{F}_λ is the following.

Definition 7.2. \mathcal{F}_λ is the set of operators (T_k) with Schwartz kernel T_k such that

- $\Pi_k T_k \Pi_k = T_k$ and $\mathcal{L}_\theta^* T_k = T_k \mathcal{L}_\theta^* = T_k$, for all $\theta \in \mathbb{T}^d$.
- T_k is $O_\infty(k^{-\infty})$ on every compact set $K \subset M^2$ such that $K \cap \Lambda = \emptyset$
- T_k satisfies assumption 7.1 for every orbifold chart $(|V|, V, G, \pi_V)$.

By adapting the proof of theorem 5.8, we define the symbol map.

Proposition 7.3. *There exists a map $\sigma : \mathcal{F}_\lambda \rightarrow C^\infty(M_r)[[\hbar]]$ which is onto and whose kernel consists of smoothing operators.*

The following theorem will be proved in the next subsection.

Theorem 7.4. $\Pi_{\lambda,k}$ is an elliptic operator of \mathcal{F}_λ .

Remark 7.5. $\Pi_{\lambda,k}$ is elliptic means that its principal symbol doesn't vanish. This implies that $\Pi_{\lambda,k}$ itself doesn't vanish when k is sufficiently large, so $\mathcal{H}_{\lambda,k}$ is not reduced to (0) when k is large enough. This completes the proof of the Guillemin-Sternberg theorem 3.3 in the orbifold case. \square

In proposition 6.5, we defined the total symbol map $\sigma : \mathcal{F}_r \rightarrow C^\infty(M_r)[[\hbar]]$ and prove that its kernel consists of smoothing operators, without using that $\Pi_{r,k} \in \mathcal{F}_r$. We can now generalize theorem 5.10.

Theorem 7.6. *The map $\mathcal{F}_\lambda \rightarrow \mathcal{F}_r$ which sends T_k into $V_k T_k V_k^*$ is well-defined and bijective. Furthermore, the total symbols of $T_k \in \mathcal{F}_\lambda$ and $V_k T_k V_k^*$ are the same.*

The proof follows the same line as the proof of theorem 5.10. To every chart $(|V|, V, G, \pi_V)$ of M is associated a chart $(|U|, U, G, \pi_U)$. Assumption 7.1 corresponds to assumption 6.2.

As a corollary of this theorem, the total symbol map $\sigma : \mathcal{F}_r \rightarrow C^\infty(M_r)[[\hbar]]$ is onto. Furthermore it follows from the stationary phase lemma that \mathcal{F}_r is a $*$ -algebra and the induced product $*_r$ on $C^\infty(M_r)[[\hbar]]$ is a star-product.

Let $\tilde{\Pi}_{r,k}$ be an operator of \mathcal{F}_r whose total symbol is the unit of $(C^\infty(M_r)[[\hbar]], *_r)$.

Lemma 7.7. $(W_k \tilde{\Pi}_{r,k} W_k^*)(V_k^* V_k) = \Pi_{\lambda,k} + R_k$ where R_k is $O(k^{-\infty})$.

Proof. Since $W_k^* V_k^* = \Pi_{r,k}$ and $V_k^* W_k^* = \Pi_{\lambda,k}$, we have

$$\begin{aligned} W_k \tilde{\Pi}_{r,k} W_k^* V_k^* V_k &= W_k \tilde{\Pi}_{r,k} V_k \\ &= W_k \tilde{\Pi}_{r,k} V_k V_k^* W_k^* \end{aligned}$$

By theorem 7.6, $V_k V_k^* = V_k \Pi_{\lambda,k} V_k^*$ belongs to \mathcal{F}_r since $\Pi_{\lambda,k}$ belongs to \mathcal{F}_λ . From the symbolic calculus of the operators of \mathcal{F}_r , it follows that

$$\tilde{\Pi}_{r,k}(V_k V_k^*) \equiv (V_k V_k^*)$$

modulo an operator of \mathcal{F}_r whose total symbol vanishes. Applying again theorem 7.6, we obtain that

$$W_k(\tilde{\Pi}_{r,k}(V_k V_k^*)) W_k^* \equiv W_k V_k V_k^* W_k^*$$

modulo an operator of \mathcal{F}_λ whose total symbol vanishes. The left-hand side is equal to $W_k \tilde{\Pi}_{r,k} W_k^* V_k^* V_k$ and the right-hand side to $\Pi_{\lambda,k}$. This proves (7.7). \square

Denote by $(V_k^* V_k)^{-1}$ the inverse of $V_k^* V_k$ on $\mathcal{H}_{\lambda,k}$, that is

$$\begin{aligned} \Pi_{\lambda,k}(V_k^* V_k)^{-1} \Pi_{\lambda,k} &= (V_k^* V_k)^{-1} \text{ and} \\ (V_k^* V_k)(V_k^* V_k)^{-1} &= (V_k^* V_k)^{-1}(V_k^* V_k) = \Pi_{\lambda,k}. \end{aligned}$$

Lemma 7.7 implies that

$$(41) \quad (V_k^* V_k)^{-1} = W_k \tilde{\Pi}_{r,k} W_k^* - R_k (V_k^* V_k)^{-1}.$$

By theorem 7.6, $W_k \tilde{\Pi}_{r,k} W_k^*$ belongs to \mathcal{F}_λ .

Lemma 7.8. $R_k(V_k^* V_k)^{-1}$ is $O(k^{-\infty})$.

Proof. Since $W_k \tilde{\Pi}_{r,k} W_k^*$ belongs to \mathcal{F}_λ , its kernel is $O(k^{n-\frac{d}{2}})$. So $W_k \tilde{\Pi}_{r,k} W_k^*$ is $O(k^{n-\frac{d}{2}})$. Using that R_k is $O(k^{-\infty})$, it follows from (41) that $(V_k^* V_k)^{-1}$ is $O(k^{n-\frac{d}{2}})$. \square

We deduce from this that $(V_k^* V_k)^{-1} \in \mathcal{F}_\lambda$. We have

$$\Pi_{r,k} = V_k (V_k^* V_k)^{-1} V_k^*.$$

Consequently theorem 7.6 implies that $\Pi_{r,k} \in \mathcal{F}_r$.

7.2. The projector $\Pi_{\lambda,k}$. This section is devoted to the proof of theorem 7.4. We use that $\Pi_k \in \mathcal{F}$ together with the following consequence of (16)

$$(42) \quad \Pi_{\lambda,k}(\underline{x}, x) = \int_{\mathbb{T}^d} ((\mathcal{L}_\theta \boxtimes \text{Id})^* \Pi_k)(\underline{x}, x) \delta_{\mathbb{T}^d}(\theta).$$

The only difficult point is to prove that $\Pi_{\lambda,k}$ satisfies assumption 7.1.

Introduce an orbifold chart $(|V|, V, G, \pi_V)$ of M as in the previous section. Denote by $\Pi_{\lambda,k,VV}$ and $\Pi_{k,VV}$ the lifts of $\Pi_{\lambda,k}$ and Π_k to V^2 . Then

$$\Pi_{\lambda,k,VV}(\underline{v}, v) = \int_{\mathbb{T}^d} ((\mathcal{L}_\theta \boxtimes \text{Id})^* \Pi_{k,VV})(\underline{v}, v) \delta_{\mathbb{T}^d}(\theta).$$

We know that $\Pi_k \in \mathcal{F}$. So $\Pi_{k,V}$ satisfies assumption 6.2 for the orbifold M . Denote by $\Pi'_{k,V}$ an associated kernel such that

$$\Pi_{k,VV} = \sum_{g \in G} (\mathcal{A}_g \boxtimes \text{Id})^* \Pi'_{k,V}.$$

If we prove that $\Pi'_{\lambda,k,V}$ given by

$$(43) \quad \Pi'_{\lambda,k,V}(\underline{v}, v) = \int_{\mathbb{T}^d} ((\mathcal{L}_\theta \boxtimes \text{Id})^* \Pi'_{k,V})(\underline{v}, v) \delta_{\mathbb{T}^d}(\theta).$$

satisfies (40), we are done. So the proof is locally reduced to the manifold case.

Let us relate the section $E_{\Delta_V}^\kappa$ and $E_{\Lambda_V}^\kappa$ appearing in (30) and (40). Let

$$s^\kappa : V \rightarrow L_V^\kappa$$

be a holomorphic $\mathbb{T}_\mathbb{C}^d$ -invariant section which doesn't vanish. Introduce the real function H such that $(s^\kappa, s^\kappa)(v) = e^{-\kappa H(v)}$ and the unitary section $t^\kappa = e^{\kappa H/2} s^\kappa$. Let us write

$$E_{\Delta_V}^\kappa(\underline{v}, v) = e^{-\kappa \phi_\Delta(\underline{v}, v)} t^\kappa(\underline{v}) \otimes t^\kappa(v), \quad E_{\Lambda_V}^\kappa(\underline{v}, v) = e^{-\kappa \phi_\Lambda(\underline{v}, v)} t^\kappa(\underline{v}) \otimes t^\kappa(v)$$

So we have

$$\Pi'_{k,V}(\underline{v}, v) = \left(\frac{k}{2\pi}\right)^n e^{-k \phi_\Delta(\underline{v}, v)} f(\underline{v}, v, k) t^k(\underline{v}) \otimes t^k(v) + O_\infty(k^{-\infty}).$$

Assumption (26.i) determines only the Taylor expansion of E_Γ^κ along Γ . Hence, the functions ϕ_Δ and ϕ_Λ are unique modulo a function which vanishes to any order along the associated Lagrangian manifold.

Recall that we introduced a function φ in proposition 4.8. Let $\tilde{\varphi}(\underline{v}, v)$ be a function such that $\tilde{\varphi}(v, v) = \varphi(v)$ and

$$\bar{Z}\tilde{\varphi} \equiv Z.\tilde{\varphi} \equiv 0$$

modulo $O(|\underline{v} - v|^\infty)$ for every holomorphic vector field Z of V . Here $\bar{Z}\tilde{\varphi}$ (resp. $Z.\tilde{\varphi}$) denote the derivative of $\tilde{\varphi}$ with respect to the vector field $(\bar{Z}, 0)$ of V^2 (resp. $(0, Z)$). We use the same notation in the following.

Lemma 7.9. *We can choose the functions ϕ_Λ and ϕ_Δ in such a way that*

$$\phi_\Delta(\underline{v}, v) = \phi_\Lambda(\underline{v}, v) - \tilde{\varphi}(\underline{v}, v)$$

Proof. Recall that $V = \mathbb{T}^d \times \mathfrak{t}_d \times U \ni (\theta, t, u) = v$. By proposition 4.8, we have

$$(44) \quad H(v) = H_r(u) + \varphi(t, u)$$

By reduction, U is endowed with a complex structure (cf. section 3.2). Introduce a function $\tilde{H}_r(\underline{u}, u)$ such that $\tilde{H}_r(u, u) = H_r(u)$ and

$$\bar{Z}\tilde{H}_r \equiv Z.\tilde{H}_r \equiv 0 \quad \text{mod } O(|\underline{u} - u|^\infty)$$

for every holomorphic vector field Z of U . Then

$$(45) \quad \phi_\Lambda(\underline{v}, v) := \frac{1}{2}(H(\underline{v}) + H(v)) - \tilde{H}_r(\underline{u}, u)$$

is a function associated to Λ_V . This is easily checked using that

$$\nabla t^\kappa = \frac{\kappa}{2}(\bar{\partial}H - \partial H) \otimes t^\kappa$$

Set $\tilde{H}(\underline{v}, v) = \tilde{H}_r(\underline{u}, u) + \tilde{\varphi}(\underline{v}, v)$. In the same way we get that

$$\phi_\Delta(\underline{v}, v) := \frac{1}{2}(H(\underline{v}) + H(v)) - \tilde{H}(\underline{v}, v)$$

is a function associated to Δ_V . □

Lemma 7.10. *We have over V^2*

$$\Pi'_{\lambda,k,V}(\underline{v}, v) = \left(\frac{k}{2\pi}\right)^{n-\frac{d}{2}} e^{-k\phi_\Lambda(\underline{v}, v)} g(\underline{v}, v, k) t^k(\underline{v}) \otimes t^k(v) + O_\infty(k^{-\infty})$$

where

$$g(\underline{v}, v, k) = \left(\frac{k}{2\pi}\right)^{\frac{d}{2}} \int_W e^{k\phi(\theta', \underline{v}, v)} f(\theta + \theta', \underline{t}, \underline{u}, v, k) |d\theta'|$$

W is any neighborhood of 0 in \mathbb{T}^d and $\phi(\theta', \underline{v}, v) = \tilde{\varphi}(\theta + \theta', \underline{t}, \underline{u}, v)$.

Proof. Since $\mathcal{L}_\theta^* t^k = t^k$, (43) implies

$$\Pi'_{\lambda,k,V}(\underline{v}, v) \equiv \left(\frac{k}{2\pi}\right)^n t^k(\underline{v}) \otimes t^k(v) \int_{\mathbb{T}^d} e^{-k\phi_\Delta(\underline{\theta} + \theta', \underline{t}, \underline{u}, v)} f(\underline{\theta} + \theta', \underline{t}, \underline{u}, v, k) |d\theta'|$$

modulo $O_\infty(k^{-\infty})$. Replacing $\theta' + \underline{\theta} - \theta$ by θ' , this leads to

$$\Pi'_{\lambda,k,V}(\underline{v}, v) \equiv \left(\frac{k}{2\pi}\right)^n t^k(\underline{v}) \otimes t^k(v) \int_{\mathbb{T}^d} e^{-k\phi_\Delta(\theta + \theta', \underline{t}, \underline{u}, v)} f(\theta + \theta', \underline{t}, \underline{u}, v, k) |d\theta'|$$

modulo $O_\infty(k^{-\infty})$. Since the imaginary part of Φ_Δ is positive outside the diagonal of V^2 , we can restrict the integral over any neighborhood W of 0 in \mathbb{T}^d . Now using lemma 7.9 and the fact that ϕ_Λ is independent of θ which appears in equation (45), we obtain the result. □

To complete the proof of theorem 7.4, it suffices to prove that $g(\cdot, k)$ admits an asymptotic expansion in power of k for the C^∞ topology on a neighborhood $\Lambda \cap V^2$. We prove this by applying the stationary phase lemma [16]. So the result is a consequence of the following lemma.

Lemma 7.11. *Let $(\underline{v}_0, v_0) \in \Lambda \cap V^2$. Then the Hessian $d_{\theta'}^2 \phi$ at $(0, \underline{v}_0, v_0)$ is a real definite positive matrix. Furthermore, on a neighborhood of $(0, \underline{v}_0, v_0)$ in $\mathbb{T}^d \times V^2$,*

$$\phi = \sum h_{ij}(\partial_{\theta'} \phi)(\partial_{\theta'} \phi)$$

where the h_{ij} are C^∞ functions of $\theta', \underline{v}, v$.

Before we prove this lemma, let us state some intermediate results. If h is a function of $C^\infty(V)$, we denote by \tilde{h} a function of $C^\infty(V^2)$ such that $\tilde{h}(\underline{v}, v) = h(v)$ and

$$(46) \quad \bar{Z}.\tilde{h} \equiv Z.\tilde{h} \equiv 0 \pmod{O(|\underline{v} - v|^\infty)}$$

for every holomorphic vector field Z of V .

Denote by t^i the coordinates of $t = \sum t^i \xi_i \in \mathfrak{t}_d$. Let us compute the derivatives of \tilde{h} with respect to the vector fields $\partial_{\theta_j} = \xi_j^\#$ and $\partial_{t_j} = J\xi_j^\#$ acting on the left and the right respectively.

Lemma 7.12. *If h is \mathbb{T}^d -invariant, then*

$$i\partial_{\underline{\theta}_j} \tilde{h} \equiv \partial_{\underline{t}_j} \tilde{h} \equiv -i\partial_{\theta_j} \tilde{h} \equiv \partial_{t_j} \tilde{h} \equiv \frac{1}{2} \tilde{h}_j \pmod{O(|\underline{v} - v|^\infty)}$$

where $h_j = \partial_{t_j} h$.

In particular, we obtain the following relations

$$(47) \quad 2\partial_{\underline{t}_j} \tilde{t}^i \equiv \delta_{ij}, \quad 2\partial_{t_j} \tilde{t}^i \equiv \delta_{ij}, \quad 2\partial_{\underline{\theta}_j} \tilde{t}^i \equiv -i\delta_{ij}, \quad 2\partial_{\theta_j} \tilde{t}^i \equiv i\delta_{ij}$$

modulo $O(|\underline{v} - v|^\infty)$.

Proof. Since $[Z, \partial_{\theta^j}]$ (resp. $[Z, \partial_{t^j}]$) is a holomorphic vector field when Z is, the various derivatives we need to compute satisfy equations (46). So we just have to compute their restriction to the diagonal of V^2 . To do this observe that

$$\langle d\tilde{h}, \partial_{\underline{\theta}^j} + i\partial_{\underline{t}^j} \rangle|_{(v,v)} = 0, \quad \langle d\tilde{h}, \partial_{\theta^j} - i\partial_{t^j} \rangle|_{(v,v)} = 0$$

since $\partial_{\theta^j} - i\partial_{t^j} = \xi_j^\# - iJ\xi_j^\#$ is holomorphic. Furthermore

$$\langle d\tilde{h}, \partial_{\underline{\theta}^j} + \partial_{\underline{t}^j} \rangle|_{(v,v)} = 0, \quad \langle d\tilde{h}, \partial_{\underline{t}^j} + \partial_{t^j} \rangle|_{(v,v)} = \partial_{t^j} \cdot h(v)$$

since $\tilde{h}(v, v) = h(v)$ and h is \mathbb{T}^d -invariant. \square

Proof of lemma 7.11. Let us write $\varphi(v) = \frac{1}{2} \sum t^i t^j \varphi_{ij}(v)$ on a neighborhood of v_0 . Consequently, we have on a neighborhood of (v_0, v_0)

$$(48) \quad \tilde{\varphi} = \frac{1}{2} \sum \tilde{t}^i \tilde{t}^j \tilde{\varphi}_{ij}.$$

It follows from equations (47) and proposition 4.10 that

$$\begin{aligned} \partial_{\underline{\theta}^i} \partial_{\underline{\theta}^j} \tilde{\varphi}(v_0, v_0) &= -\varphi_{ij}(v_0) \\ &= -2g(\xi_i^\#, \xi_j^\#)(v_0). \end{aligned}$$

The first part of the lemma follows. By (48) there exists C^∞ functions h_{ij}^1 such that $\partial_{\underline{\theta}^i} \tilde{\varphi} = \sum \tilde{t}^j h_{ij}^1$ on a neighborhood of (v_0, v_0) . Derivating with respect to $\underline{\theta}$, we obtain

$$h_{ij}^1(v_0, v_0) = -i\varphi_{ij}(v_0).$$

Hence it is an invertible matrix and there exists functions h_{ij}^2 such that $\tilde{t}^i = \sum h_{ij}^2 \partial_{\underline{\theta}^j} \tilde{\varphi}$ on a neighborhood of (v_0, v_0) . By (48),

$$\tilde{\varphi} = \sum h_{ij}^3 (\partial_{\underline{\theta}^i} \tilde{\varphi})(\partial_{\underline{\theta}^j} \tilde{\varphi})$$

for some C^∞ functions h_{ij}^3 on a neighborhood of (v_0, v_0) . The second part of the lemma follows. \square

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